Note on Nordhaus-Gaddum problems for power domination

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Abstract

The upper and lower Nordhaus-Gaddum bounds over all graphs for the power domination number follow from known bounds on the domination number and examples. In this note we improve the upper sum bound for the power domination number substantially for graphs having the property that both the graph and its complement must be connected. For these graphs, our bound is tight and is also significantly better than the corresponding bound for domination number. We also improve the product upper bound for the power domination number for graphs with certain properties.

Keywords: power domination; domination; zero forcing; Nordhaus-Gaddum

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1 Introduction

The study of the power domination number of a graph arose from the question of how to monitor electric power networks at minimum cost, see Haynes et al. [9]. Intuitively, the power domination problem consists of finding a set of vertices in a graph that can observe the entire graph according to certain observation rules. The formal definition is given below immediately after some graph theory terminology.

A *graph* $G = (V, E)$ is an ordered pair formed by a finite nonempty set of *vertices* $V = V(G)$ and a set of *edges* $E = E(G)$ containing unordered pairs of distinct vertices (that is, all graphs are simple and undirected). The *complement* of $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ consists of all two element subsets of V that are not in E. For any vertex $v \in V$, the *neighborhood* of v is the set $N(v) = \{u \in V : \{u, v\} \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. Similarly, for any set of vertices S, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$.

For a set S of vertices in a graph G, define $PD(S) \subseteq V(G)$ recursively as follows:

1. $PD(S) := N[S] = S \cup N(S)$.

2. While there exists $v \in PD(S)$ such that $|N(v) \setminus PD(S)| = 1$: $PD(S) := PD(S) \cup N(v)$.

A set $S \subseteq V(G)$ is called a *power dominating set* of a graph G if, at the end of the process above, $PD(S) = V(G)$. A *minimum power dominating set* is a power dominating set of minimum cardinality. The *power domination number* of G, denoted by $\gamma_P(G)$, is the cardinality of a minimum power dominating set.

Power domination is naturally related to domination and to zero forcing. A set $S \subseteq V(G)$ is called a *dominating set* of a graph G if $N[S] = V(G)$. A *minimum dominating set* is a dominating set of minimum cardinality. The *domination number* of G, denoted by $\gamma(G)$, is the cardinality of a minimum dominating set. Clearly $\gamma_P(G) \leqslant \gamma(G)$.

Zero forcing was introduced independently in combinatorial matrix theory [1] and control of quantum systems [5]. From a graph theory point of view, zero forcing is a coloring game on a graph played according to the *color change rule*: If u is a blue vertex and exactly one neighbor w of u is white, then change the color of w to blue. We say u *forces* w. A *zero forcing set* for G is a subset of vertices B such that when the vertices in B are colored blue and the remaining vertices are colored white initially, repeated application of the color change rule can color all vertices of G blue. A *minimum zero forcing set* is a zero forcing set of minimum cardinality. The *zero forcing number* of G, denoted by $\mathrm{Z}(G)$, is the cardinality of a minimum zero forcing set. Power domination can be seen as a domination step followed by a zero forcing process, and we will use the terminology "v forces w" to refer to Step 2 of power domination. Clearly $\gamma_P(G) \leqslant \mathrm{Z}(G)$.

For a graph parameter ζ, the following are *Nordhaus-Gaddum* problems:

- Determine a (tight) lower or upper bound on $\zeta(G) + \zeta(\overline{G})$.

- Determine a (tight) lower or upper bound on $\zeta(G) \cdot \zeta(\overline{G})$.

The name comes from the next theorem of Nordhaus and Gaddum, where $\chi(G)$ denotes the chromatic number of G.

Theorem 1. [16] *For any graph G of order n,*

$$2\sqrt{n} \leqslant \chi(G) + \chi(\overline{G}) \leqslant n + 1$$

and

$$n \leqslant \chi(G) \cdot \chi(\overline{G}) \leqslant \left(\frac{n+1}{2}\right)^2.$$

Each bound is assumed for infinitely many values of n.

Nordhaus-Gaddum bounds have been found for both domination and zero forcing. In addition to the original papers cited here, Nordhaus-Gaddum results for domination and several variants (but not power domination) are discussed in Section 9.1 of the book [10] and in the survey paper [2].

Theorem 2. [13] *For any graph G of order $n \geqslant 2$,*

$$3 \leqslant \gamma(G) + \gamma(\overline{G}) \leqslant n + 1 \qquad \text{and} \qquad 2 \leqslant \gamma(G) \cdot \gamma(\overline{G}) \leqslant n.$$

The upper bounds are realized by the complete graph K_n, and the lower bounds are realized by the star (complete bipartite graph) $K_{1,n-1}$.

It is known that for a graph G of order $n \geqslant 2$,

$$n - 2 \leqslant \mathrm{Z}(G) + \mathrm{Z}(\overline{G}) \leqslant 2n - 1$$

and

$$n - 3 \leqslant \mathrm{Z}(G) \cdot \mathrm{Z}(\overline{G}) \leqslant n^2 - n,$$

with the upper bounds realized by the complete graph K_n and the lower bounds realized by the path P_n for $n \geqslant 4$. That the upper bounds are correct is immediate. The result $n-2 \leqslant \mathrm{Z}(G)+\mathrm{Z}(\overline{G})$ appears in [7]. Then $n-3 \leqslant \mathrm{Z}(G)\cdot\mathrm{Z}(\overline{G})$ follows, because $1 \leqslant \mathrm{Z}(G)$ for all G and the function $f(z) = z(n-2-z)$ attains its minimum on the interval $[1, n-3]$ at the endpoints.

The general Nordhaus-Gaddum upper bounds for power domination number follow from those for domination number given in Theorem 2. The inequalities $2 \leqslant \gamma_P(G) + \gamma_P(\overline{G})$ and $1 \leqslant \gamma_P(G)\cdot\gamma_P(\overline{G})$ are obvious since $1 \leqslant \gamma_P(G)$ for every graph, and these are realized by the path P_n (it is straightforward to verify that $\gamma_P(P_n) = 1 = \gamma_P(\overline{P_n})$).

Corollary 3. *For any graph G of order n,*

$$2 \leqslant \gamma_P(G) + \gamma_P(\overline{G}) \leqslant n + 1 \qquad \text{and} \qquad 1 \leqslant \gamma_P(G) \cdot \gamma_P(\overline{G}) \leqslant n.$$

The upper bounds are realized by the complete graph K_n, and the lower bounds are realized by the path P_n.

In Section 3 we improve the sum upper bound for the power domination number significantly under the assumption that both G and $\overline{G}$ are connected, or more generally all components of both have order at least 3, and show that this bound is substantially different from the analogous bound for domination number. In Section 4 we refine the product bounds for certain special cases. Section 2 contains additional results that we use in Sections 3 and 4. Section 5 summarizes the bounds for domination number, power domination number, and zero forcing number.

Some additional notation is used: Let $K_{p,q}$ denote a complete bipartite graph with partite sets of cardinality p and q. The *degree* of vertex v is $\deg_G v = |N_G(v)|$. Let $\delta(G)$ (respectively, $\Delta(G)$) denote the minimum (respectively, maximum) of the degrees of the vertices of G. A *cut-set* is a set of vertices whose removal disconnects G. The *vertex-connectivity* of $G \neq K_n$, denoted by $\kappa(G)$, is the minimum cardinality of a cut-set (note $\kappa(G) = 0$ if G is disconnected), and $\kappa(K_n) = n - 1$. An *edge-cut* is a set of edges whose removal disconnects G, and the *edge-connectivity* of G, denoted by $\lambda(G)$, is the minimum cardinality of an edge-cut. Observe that $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$. The *distance* between vertices u and v in G, $d_G(u, v)$, is the length of a shortest path between u and v in G. The *diameter* of G, $\operatorname{diam}(G)$, is the maximum distance between two vertices in a connected graph G; $\operatorname{diam}(G) = \infty$ if G is not connected. A *component* of a graph is a maximal connected subgraph.

2 Tools for Nordhaus-Gaddum bounds for power domination

In this section we establish results that will be applied to improve Nordhaus-Gaddum upper bounds for both the sum and product of the power domination number with additional assumptions, such as every component of the graph and its complement has order at least 3. The next result is immediate from Corollary 3.

Corollary 4. *For any graph G of order n, $\gamma_P(G) \leqslant \left\lfloor \frac{n}{\gamma_P(\overline{G})} \right\rfloor$.*

Next we consider the relationship between the power domination number of G or $\overline{G}$ and the minimum degree or vertex-connectivity of G.

Remark 5. For any graph G of order n, $\gamma(\overline{G}) \leqslant \delta(G) + 1$, because a vertex of maximum degree in $\overline{G}$, which is $n - 1 - \delta(G)$, together with all its non-neighbors is a dominating set of $\overline{G}$.

Proposition 6. *Let G be a graph such that neither G nor $\overline{G}$ has isolated vertices. Then $\gamma_P(\overline{G}) \leqslant \delta(G)$. If $\delta(G) = 1$, then $\gamma_P(\overline{G}) = 1$.*

Proof. Construct a power dominating set S for $\overline{G}$ of cardinality $\delta(G)$ as follows: Put a vertex v of maximum degree in $\overline{G}$ into S, so $|N_{\overline{G}}[v]| = \Delta(\overline{G}) + 1 = n - 1 - \delta(G) + 1 = n - \delta(G) < n$, where n is the order of G. Then add all but one of the vertices in $V(\overline{G}) \setminus N_{\overline{G}}[v]$ into S, i.e., add $\delta(G) - 1 \geqslant 0$ vertices to S, so $|S| = \delta(G)$. Now $N_{\overline{G}}[S]$ contains all but at most one vertex, and since $\overline{G}$ has no isolated vertices, any neighbor of such a vertex can force it. The last statement then follows since $\gamma_P(G) \geqslant 1$ for all graphs G. □

Theorem 7. [11] *If G is a graph with* $\operatorname{diam}(G) = 2$*, then* $\gamma(G) \leqslant \kappa(G)$*.*

Next we state several results that give sufficient conditions for $\gamma(G) \leqslant 2$ or $\gamma(\overline{G}) \leqslant 2$, which then imply $\gamma_P(G) \leqslant 2$ or $\gamma_P(\overline{G}) \leqslant 2$.

Theorem 8. [4], [10, Theorem 2.25] *If G is a graph with* $\operatorname{diam}(G) \geqslant 3$*, then* $\gamma(\overline{G}) \leqslant 2$*.*

Note that Theorem 8 also applies to graphs that are not connected.

Theorem 9. *Suppose G is a graph with* $\operatorname{diam}(G) = 2$ *such that $\overline{G}$ has no isolated vertices. Then $\gamma_P(G) \leqslant \kappa(G) - 1$ or $\gamma_P(\overline{G}) \leqslant 2$.*

Proof. Since $\overline{G}$ has no isolated vertices, every vertex has a neighbor in $\overline{G}$. Let S be a minimum cut-set for G. Since $\operatorname{diam}(G) = 2$, every vertex in $V \setminus S$ is adjacent to at least one vertex in S.

Case 1: There exists a vertex $u \in V \setminus S$ that is adjacent to exactly one vertex in S, say v (Case 1 is the only possible case when $\kappa(G) = 1$). Let G_1 denote the component of $G - S$ containing u. In $\overline{G}$, u dominates $S \setminus \{v\}$ and all vertices in components of $G - S$ other than G_1. Let x be any vertex in a component of $G - S$ that is not equal to G_1. Then x dominates the vertices of G_1. Therefore, $\{u, x\}$ dominates all vertices in V except possibly v, and any neighbor of v in $\overline{G}$ can force v, so $\{u, x\}$ is a power dominating set for $\overline{G}$. Thus, $\gamma_P(\overline{G}) \leqslant 2$.

Case 2: Every vertex in $V \setminus S$ is adjacent to at least two vertices in S. Then $S \setminus \{v\}$ is a power dominating set for any vertex $v \in S$, because $S \setminus \{v\}$ dominates $V \setminus \{v\}$, and any neighbor of v in G can force v. Thus, $\gamma_P(G) \leqslant \kappa(G) - 1$. □

Theorem 10. [8] *If G is planar and* $\operatorname{diam}(G) = 2$*, then $\gamma(G) \leqslant 2$ or $G = S_4(K_3)$, the graph shown in Figure 1. Furthermore, $\gamma(S_4(K_3)) = 3$.*

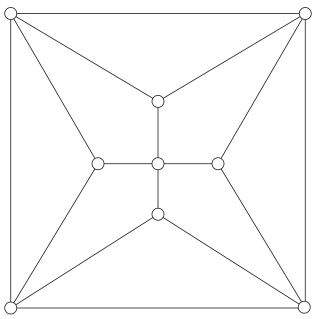

Figure 1: The graph $S_4(K_3)$, which is the only planar graph with diameter 2 and domination number greater than 2.

Corollary 11. *If G is planar and* $\operatorname{diam}(G) = 2$*, then $\gamma_P(G) \leqslant 2$.*

Proof. This follows from Theorem 10 and the fact that $\gamma_P(S_4(K_3)) = 2$. □

When $\kappa(G) = \lambda(G) = \delta(G)$, G is *maximally connected*. In every maximally connected graph G, for any vertex v such that $\deg v = \delta(G)$, $N_G(v)$ is a minimum cut-set and the set of all edges incident with v is a minimum edge-cut. In this case we say the cut is *trivial*, because it leaves a connected component formed by one isolated vertex. A maximally connected graph G is *super-λ* if every minimum edge-cut is trivial. Super-λ graphs of diameter 2 were characterized by Wang and Li:

Theorem 12. [19] *A connected graph G with $\mathrm{diam}(G) = 2$ is super-λ if and only if G contains no subgraph $K_{\delta(G)}$ in which all of the vertices have degree (in G) equal to $\delta(G)$.*

Proposition 13. *Let G be a connected graph with $\mathrm{diam}(G) = 2$. If G is not super-λ, then $\gamma_P(G) \leqslant 2$.*

Proof. Since G is not super-λ, there exists a subgraph $K_{\delta(G)}$ in which all of the vertices have degree equal to $\delta(G)$ in G. Let v be a vertex in this $K_{\delta(G)}$, so v has exactly one neighbor outside $K_{\delta(G)}$, say w. Then, v dominates all vertices in $K_{\delta(G)}$ and w. Since every vertex u in $K_{\delta(G)}$ has degree in G equal to $\delta(G)$ and u has $\delta(G)-1$ dominated neighbors, u can force its one remaining neighbor. Therefore, all vertices in $K_{\delta(G)}$ and their neighbors are observed. Since $\mathrm{diam}(G) = 2$, $d(v, x) = 1$ or $d(v, x) = 2$ for every vertex $x \neq v$ in G. If $d(v, x) = 1$, then x is dominated by v. If $d(v, x) = 2$, then x is a neighbor of a vertex in $N_G(v)$. Since the vertices in $N_G(v)$ that are in $K_{\delta(G)}$ have forced their neighbors, the only case in which x is not observed is if it is a neighbor of w. Thus $\{v, w\}$ is a power dominating set. □

Corollary 14. *Assume that G and $\overline{G}$ both have all components of order at least 3. Then $\gamma_P(G) \leqslant 2$ or $\gamma_P(\overline{G}) \leqslant 2$ if any of the conditions below is satisfied:*

1. $\mathrm{diam}(G) \geqslant 3$ *or* $\mathrm{diam}(\overline{G}) \geqslant 3$.
2. *G or $\overline{G}$ is planar.*
3. $\kappa(G) \leqslant 3$ *or* $\kappa(\overline{G}) \leqslant 3$.
4. *G or $\overline{G}$ is not super-λ.*

Proof. Part (1) follows from Theorem 8. Since G and $\overline{G}$ both have all components of order at least 3, $\mathrm{diam}(G) \neq 1$ and $\mathrm{diam}(\overline{G}) \neq 1$. The case $\mathrm{diam}(G) \geqslant 3$ is covered by part (1). So assume $\mathrm{diam}(G) = 2$. Then (2), (3), and (4) follow from Corollary 11, Theorem 9, and Proposition 13, respectively. □

Let $\mathcal{T}$ be the family of graphs constructed by starting with a connected graph H and for each $v \in V(H)$ adding two new vertices v' and v'', each adjacent to v and possibly to each other but not to any other vertices. The next result appears in [21] without the floor function.

Theorem 15. [21] *Suppose every component of a graph G has order at least 3 and n denotes the order of G. Then $\gamma_P(G) \leqslant \left\lfloor \frac{n}{3} \right\rfloor$. Furthermore, if $\gamma_P(G) = \frac{n}{3}$, then every component of G is in $\mathcal{T} \cup \{K_{3,3}\}$.*

The method used in the construction of a graph $G \in \mathcal{T}$ implies that $\gamma_P(\overline{G}) = 1$ if we start with a graph on at least 2 vertices:

Lemma 16. *Suppose G is a graph having distinct vertices w, u, v, v', and v'' such that $N[v'] = N[v''] = \{v, v', v''\}$, $u \in N(v)$ and $w \notin N(v)$. Then $\gamma_P(\overline{G}) = 1$.*

Proof. In $\overline{G}$, u is not adjacent to v but is adjacent to v' and to v''. Then $\{v'\}$ is a power dominating set for $\overline{G}$, because $u \in N_{\overline{G}}[v'] = V(G) \setminus \{v, v''\}$ and u forces v'' in $\overline{G}$, and then w forces v in $\overline{G}$. Thus $\gamma_p(\overline{G}) = 1$. □

Proposition 17. *Suppose G is a graph of order n such that every component of G and $\overline{G}$ has order at least 3 and $\gamma_P(G) = \frac{n}{3}$. Then $\gamma_P(\overline{G}) \leqslant 2$. If, in addition, G has a component $G_o \in \mathcal{T}$ of order at least 6, then $\gamma_P(\overline{G}) = 1$.*

Proof. Necessarily, n is a multiple of 3 and $n \neq 3$. If G has 2 or more components, then $\gamma(\overline{G}) \leqslant 2$ by Theorem 8. If $G = K_{3,3}$, then $\gamma_P(\overline{G}) = 2$. Now suppose G has a component $G_o \in \mathcal{T}$ of order at least 6 (this includes the case where G has only one component that is not $K_{3,3}$). Then $\gamma_P(\overline{G_o}) = 1$ by Lemma 16 and Proposition 6 (for the case $v'' \notin N(v')$). In $\overline{G}$, any vertex in G_o dominates any vertex in a different component, so the one vertex that power dominates $\overline{G_o}$ also power dominates $\overline{G}$, and $\gamma_P(\overline{G}) = 1$. □

Theorem 18. [11, 14] *Suppose G is a graph of order n with* $\text{diam}(G) = 2$. *If $n \geqslant 24$, then $\gamma(G) \leqslant \lfloor \frac{n}{4} \rfloor$, and $\gamma(G) \leqslant \lfloor \frac{n}{4} \rfloor + 1$ for $n \leqslant 23$.*

Remark 19. Let G be a graph. Suppose W is a set of at least two vertices such that no vertex outside W is adjacent to exactly one vertex in W. Then every power dominating set S must contain a vertex in $N[W]$, because no vertex outside of W can force a vertex in W unless all but one of the vertices in W have already been power dominated.

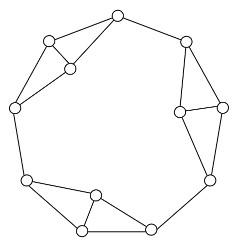

Figure 2: The necklace N_3

The *necklace* with s diamonds, denoted N_s, is a 3-regular graph that can be constructed from a $3s$-cycle by appending s additional vertices, with each new vertex adjacent to 3 sequential cycle vertices; N_3 is shown in Figure 2.

Theorem 20. [6] *Suppose G is a connected 3-regular graph of order n and $G \neq K_{3,3}$. Then $\gamma_P(G) \leqslant \lfloor \frac{n}{4} \rfloor$, and this bound is attained for arbitrarily large n by $G = N_r$.*

Lemma 21. *For $r \geqslant 2$, $\gamma_P(\overline{N_r}) = 2$.*

Proof. Any two vertices that are in different copies of $K_4 - e$ and are not incident to the missing edges dominate $\overline{N_r}$, so $\gamma_P(\overline{N_r}) \leqslant 2$. To complete the proof, we show that no one vertex v can power dominate $\overline{N_r}$. Denote the vertices of the $K_4 - e$ that contains v by x, y, z, w, where $e = \{x, y\}$. Apply Remark 19 to $W = \{z, w\}$ for $v = x$ and to $W = \{x, y, w\}$ for $v = z$ to conclude $\{v\}$ is not a power dominating set; the cases $v = y$ or w are similar. □

3 Nordhaus-Gaddum sum bounds for power domination

In this section, we improve the tight Nordhaus-Gaddum sum upper bound of n for all graphs (Corollary 3) to approximately $\frac{n}{3}$ under one of the assumptions that each component of G and $\overline{G}$ has order at least 3 (Theorem 23 below), or that both G and $\overline{G}$ are connected (Theorem 25 below), and to approximately $\frac{n}{4}$ in some special cases. The lower bound $2 \leqslant \gamma_P(G) + \gamma_P(\overline{G})$ can be attained with both G and $\overline{G}$ connected, specifically by the path $G = P_n$ (both P_n and $\overline{P_n}$ are connected for $n \geqslant 4$). But the upper bound for all graphs is attainable only by disconnecting G or $\overline{G}$ with some very small components.

The next result follows from Corollary 14 and Theorem 15.

Corollary 22. *Let G be a graph of order n such that every component of G and $\overline{G}$ has order at least 3 and ($\operatorname{diam}(G) \geqslant 3$ or $\operatorname{diam}(\overline{G}) \geqslant 3$ or $\kappa(G) \leqslant 3$ or $\kappa(\overline{G}) \leqslant 3$). Then $\gamma_P(G) + \gamma_P(\overline{G}) \leqslant \lfloor \frac{n}{3} \rfloor + 2$.*

Theorem 23. *Suppose G is a graph of order n such that every component of G and $\overline{G}$ has order at least 3. Then for $n \neq 13, 14, 16, 17, 20$,*

$$\gamma_P(G) + \gamma_P(\overline{G}) \leqslant \left\lfloor \frac{n}{3} \right\rfloor + 2,$$

and this bound is attained for arbitrarily large n by $G = rK_3$ (where $r \geqslant 2$).
For $n = 13, 14, 16, 17, 20$, $\gamma_P(G) + \gamma_P(\overline{G}) \leqslant \lfloor \frac{n}{3} \rfloor + 3$.

Proof. Without loss of generality, we assume $\gamma_P(G) \leqslant \gamma_P(\overline{G})$, and let $p = \gamma_P(G)$ and $\bar{p} = \gamma_P(\overline{G})$. If $p \leqslant 2$, then $p + \bar{p} \leqslant \lfloor \frac{n}{3} \rfloor + 2$ follows from Theorem 15. If $p \geqslant 6$, Corollary 4 gives $p + \bar{p} \leqslant \frac{n}{\bar{p}} + \frac{n}{p} \leqslant \frac{n}{3}$. So we assume $3 \leqslant p \leqslant 5$. Since $\operatorname{diam}(G), \operatorname{diam}(\overline{G}) \neq 1$, by Corollary 22 we may also assume $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 2$ and $\kappa(G), \kappa(\overline{G}) \geqslant 4$. The latter implies $n \geqslant 9$. Corollary 4 implies $p + \bar{p} \leqslant p + \left\lfloor \frac{n}{p} \right\rfloor$. By Theorem 18, $p, \bar{p} \leqslant \lfloor \frac{n}{4} \rfloor + 1$. Thus we need to consider the following cases:

- $p = 3, 4$, in which case $p + \bar{p} \leqslant \lfloor \frac{n}{4} \rfloor + 4$.
- $p = 5$, in which case $p + \bar{p} \leqslant \lfloor \frac{n}{5} \rfloor + 5$.

Algebra shows that $\lfloor \frac{n}{4} \rfloor + 4 \leqslant \lfloor \frac{n}{3} \rfloor + 2$ and $\lfloor \frac{n}{5} \rfloor + 5 \leqslant \lfloor \frac{n}{3} \rfloor + 2$ for $n \geqslant 21$ and $n = 18, 19$. For $n = 9, 10, 11$, $p + \bar{p} \leqslant 5 = \lfloor \frac{n}{3} \rfloor + 2$ has been verified computationally [12].

To complete the proof that $p + \bar{p} \leqslant \frac{n}{3} + 2$ for $n \neq 13, 14, 16, 17, 20$, we consider $n = 12, 15$. Since $p \leqslant \bar{p} \leqslant \frac{n}{p}$, the only possibilities are $n = 12$ with $(p, \bar{p}) = (3,3), (3,4)$, or $n = 15$ with $(p, \bar{p}) = (3,3), (3,4), (3,5)$. For $n = 12$ with $(p, \bar{p}) = (3,3)$, and $n = 15$ with $(p, \bar{p}) = (3,3), (3,4)$, $\gamma_P(G) + \gamma_P(\overline{G}) \leqslant \frac{n}{3} + 2$. In each of the remaining cases, $n = 12$ with $(p, \bar{p}) = (3,4)$, or $n = 15$ with $(p, \bar{p}) = (3,5)$, observe that $\bar{p} = \frac{n}{3}$ and $p = 3$. But this is prohibited by Proposition 17.

If G is a disjoint union of $r \geqslant 2$ copies of K_3, then $\gamma_P(G) + \gamma_P(\overline{G}) = \frac{n}{3} + 2$, so the bound is tight for arbitrarily large n.

Finally, consider $n = 13, 14, 16, 17, 20$. For $p = 3$, $\gamma_P(G) + \gamma_P(\overline{G}) \leqslant \lfloor \frac{n}{3} \rfloor + 3$ is immediate from Theorem 15. Since $p \leqslant \bar{p} \leqslant \frac{n}{p}$, the only remaining cases are $n = 16$ or 17 with $(p, \bar{p}) = (4,4)$, and $n = 20$ with $(p, \bar{p}) = (4,4), (4,5)$. All of these satisfy $\gamma_P(G) + \gamma_P(\overline{G}) \leqslant \lfloor \frac{n}{3} \rfloor + 3$. □

We have no examples contradicting $\gamma_P(G) + \gamma_P(\overline{G}) \leqslant \lfloor \frac{n}{3} \rfloor + 2$ for graphs G of any order n where the order of each component of G and $\overline{G}$ is at least 3. We conjecture that these "exceptional values" 13, 14, 16, 17, 20 of n are not in fact exceptions:

Conjecture 24. If G is graph of order n such that the order of each component of G and $\overline{G}$ is at least 3, then $\gamma_P(G) + \gamma_P(\overline{G}) \leqslant \lfloor \frac{n}{3} \rfloor + 2$.

Next we consider the case in which both G and $\overline{G}$ are required to be connected.

Theorem 25. *Suppose G is a graph of order n such that both G and $\overline{G}$ are connected. Then for $n \neq 12, 13, 14, 15, 16, 17, 18, 20, 21, 24$,*

$$\gamma_P(G) + \gamma_P(\overline{G}) \leqslant \left\lceil \frac{n}{3} \right\rceil + 1,$$

and this bound is attained for arbitrarily large $n \geqslant 6$ by $G \in \mathcal{T}$.

Proof. For n not a multiple of 3, $\lceil \frac{n}{3} \rceil + 1 = \lfloor \frac{n}{3} \rfloor + 2$, and the result follows from Theorem 23. So assume n is a multiple of 3. We proceed as in the proof of Theorem 23, with the same notational conventions $p := \gamma_P(G) \leqslant \bar{p} := \gamma_P(\overline{G})$, and again the bound is established for $p \geqslant 6$ by Corollary 4. The result follows from Theorem 15 for $p = 1$, so we assume $2 \leqslant p \leqslant 5$. Because n is a multiple of 3 and both G and $\overline{G}$ are connected, $n \geqslant 6$. Since $\overline{K_{3,3}}$ is not connected, Theorem 15 and Proposition 17 prohibit $p \geqslant 2$ and $\bar{p} = \frac{n}{3}$, so $\bar{p} \leqslant \frac{n}{3} - 1$. Thus the result follows for $p = 2$.

So we assume $3 \leqslant p \leqslant 5$ and $3 \leqslant p \leqslant \bar{p} \leqslant \frac{n}{3} - 1$; the latter requires $n \geqslant 12$, and since all multiples of 3 between 12 and 26 are excluded by hypothesis, we may assume $n \geqslant 27$. Since $3 \leqslant p$, $\operatorname{diam}(\overline{G}) \leqslant 2$ by Theorem 8. Since $\operatorname{diam}(\overline{G}) = 1$ would imply $\bar{p} = 1$, necessarily $\operatorname{diam}(\overline{G}) = 2$. Then $\bar{p} \leqslant \lfloor \frac{n}{4} \rfloor$ by Theorem 18. Hence $p + \bar{p} \leqslant \lfloor \frac{n}{4} \rfloor + 4$ for $p = 3, 4$, and $p + \bar{p} \leqslant \lfloor \frac{n}{5} \rfloor + 5$ for $p = 5$. Algebra shows that $\lfloor \frac{n}{4} \rfloor + 4 \leqslant \frac{n}{3} + 1$ and $\lfloor \frac{n}{5} \rfloor + 5 \leqslant \frac{n}{3} + 1$ for $n \geqslant 27$ and n a multiple of 3.

There are graphs $G \in \mathcal{T}$ of arbitrarily large order n, $\overline{G}$ is connected for $n \geqslant 6$, and these graphs attain the bound. □

The tight upper bound in Theorem 25 for $\gamma_P(G)+\gamma_P(\overline{G})$ with both G and $\overline{G}$ connected was obtained by switching from floor to ceiling. This raises a question about the bound with floor, which has implications for products (see Section 4).

Question 26. Do there exist graphs G of arbitrarily large order n with both G and $\overline{G}$ connected such that $\gamma_P(G) + \gamma_P(\overline{G}) = \lfloor \frac{n}{3} \rfloor + 2$?

The next two examples, found via the computer program *Sage*, show that there are pairs of connected graphs G and $\overline{G}$ of orders $n = 8$ and 11 such that $\gamma_P(G) + \gamma_P(\overline{G}) = \lfloor \frac{n}{3} \rfloor + 2$.

Example 27. Let G be the graph shown with its complement in Figure 3; observe that both are connected. It is easy to see that no one vertex power dominates either G or $\overline{G}$ and also easy to find a power dominating set of two vertices for each. Thus

$$\gamma_P(G) + \gamma_P(\overline{G}) = 2 + 2 = 4 = \left\lfloor \frac{8}{3} \right\rfloor + 2.$$

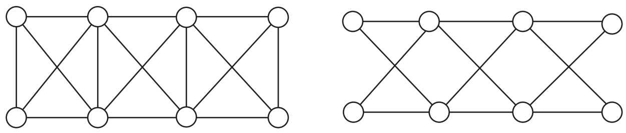

Figure 3: A connected graph G of order 8 and its connected complement $\overline{G}$ such that $\gamma_P(G) + \gamma_P(\overline{G}) = \lfloor \frac{n}{3} \rfloor + 2$.

Example 28. Let G be the graph shown in Figure 4. It is easy to see that $\overline{G}$ is also connected.

First we show that no set of two vertices is a power dominating set for G. Since $\{1, 2, 7\}$ is a power dominating set for G, this will imply $\gamma_P(G) = 3 = \lfloor \frac{11}{3} \rfloor$. By Remark 19 applied to the sets $W_1 = \{2, 3\}$ and $W_2 = \{7, 8\}$, any power dominating set S of G must contain vertices $u_1 \in \{2, 3, 4, 5, 6\}$ and analogously, $u_2 \in \{7, 8, 9, 10, 11\}$. If $u_1 \in \{2, 3\}$ and $u_2 \in \{7, 8\}$, then vertex 1 cannot be forced. If $u_1 \in \{4, 5, 6\}$, then the two remaining vertices in $\{4, 5, 6\}$ cannot be forced; the case in which $u_2 \in \{9, 10, 11\}$ is symmetric.

Next we show that no one vertex is a power dominating set for $\overline{G}$. Since $\{2, 7\}$ is a power dominating set for $\overline{G}$, this will imply $\gamma_P(\overline{G}) = 2$ and $\gamma_P(G) + \gamma_P(\overline{G}) = \lfloor \frac{11}{3} \rfloor + 2$. For each possible vertex $v \in \{1, 2, 3, 4, 5, 6\}$, we apply Remark 19 with W as shown: For $v \in \{1, 2, 3\}$, use $W = \{4, 5, 6\}$. For $v \in \{4, 5, 6\}$, use $W = \{2, 3\}$. The case $v \in \{7, 8, 9, 10, 11\}$ is symmetric.

The next two theorems for domination number provide an interesting comparison.

Theorem 29. [3] *For any graph G of order n such that $\delta(G) \geqslant 1$ and $\delta(\overline{G}) \geqslant 1$,*

$$\gamma(G) + \gamma(\overline{G}) \leqslant \left\lfloor \frac{n}{2} \right\rfloor + 2,$$

and this bound is attained for arbitrarily large n.

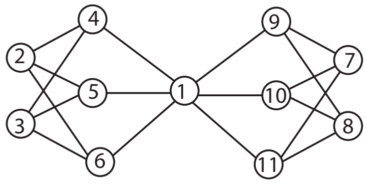

Figure 4: A connected graph G of order 11 such that $\overline{G}$ is also connected and $\gamma_P(G) + \gamma_P(\overline{G}) = \lfloor \frac{n}{3} \rfloor + 2$.

Theorem 30. [11] *Suppose G is a graph of order n such that $\delta(G) \geqslant 7$ and $\delta(\overline{G}) \geqslant 7$. Then*

$$\gamma(G) + \gamma(\overline{G}) \leqslant \left\lfloor \frac{n}{3} \right\rfloor + 2.$$

From Theorem 30 we see that the same sum upper bound we obtained for power domination number (with the weaker hypothesis that every component has order at least 3) is obtained for domination number when we make the stronger assumption that the minimum degrees of both G and $\overline{G}$ are at least 7. Theorem 29 is a more direct parallel to Theorem 23 but with a higher bound. Theorem 29 has a weaker hypothesis, which is equivalent to "every component of G and $\overline{G}$ has order at least 2." The next example shows that if Theorem 29 is restated to require both G and $\overline{G}$ to be connected, the bound remains tight. This provides a direct comparison with Theorem 25 and shows that for graphs G with both G and $\overline{G}$ connected, the upper bound for the domination sum is substantially higher than the upper bound for the power domination sum.

Example 31. Let G_k denote the kth comb, constructed by adding a leaf to every vertex of a path P_k (G_9 is shown in Figure 5); the order of G_k is $2k$. Then every dominating set S must have at least k elements, because for each of the k leaves, either the leaf or its neighbor must be in S. Since two vertices are needed to dominate $\overline{G_k}$, $\gamma(G_k) + \gamma(\overline{G_k}) = k + 2 = \frac{2k}{2} + 2$. The results for power domination are very different. For $k = 3s$, one third of the vertices in P_k can power dominate G_k, and one vertex can power dominate $\overline{G_k}$, so $\gamma_P(G_k) + \gamma_P(\overline{G_k}) = s + 1 = \frac{2k}{6} + 1$.

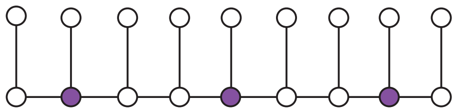

Figure 5: The comb G_9 with the vertices of a minimum power dominating set colored.

We can also improve the bound in Corollary 3 when G has some components of order less than 3 and G has at least one edge.

Theorem 32. *Let G be a graph of order n that has n_1 isolated vertices and n_2 copies of K_2 as components such that n_1 and n_2 are not both zero. Then*

$$\gamma_P(G) + \gamma_P(\overline{G}) \leqslant 1 + \frac{n}{3} + \frac{2n_1}{3} + \frac{n_2}{3}.$$

Proof. As a consequence of Theorem 15,

$$\gamma_P(G) \leqslant n_1 + n_2 + \left(\frac{n - n_1 - 2n_2}{3}\right).$$

Because $n_1 \geqslant 1$ or $n_2 \geqslant 1$, an isolated vertex (respectively, one of the vertices in a K_2 component) power dominates the complement, so $\gamma_P(\overline{G}) = 1$. Hence,

$$\gamma_P(G) + \gamma_P(\overline{G}) \leqslant n_1 + n_2 + \left(\frac{n - n_1 - 2n_2}{3}\right) + 1. \qquad \square$$

We can also improve the upper bound in some special cases.

Theorem 33. *Suppose G is a graph of order n with* $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 2$*, and one of the following is true:*

1. *G or $\overline{G}$ is planar.*
2. *$\kappa(G) \leqslant 3$ or $\kappa(\overline{G}) \leqslant 3$.*
3. *G or $\overline{G}$ is not super-λ.*

If $n \geqslant 24$, then $\gamma_P(G) + \gamma_P(\overline{G}) \leqslant \lfloor \frac{n}{4} \rfloor + 2$, and $\gamma_P(G) + \gamma_P(\overline{G}) \leqslant \lfloor \frac{n}{4} \rfloor + 3$ for $n \leqslant 23$.

Proof. By Corollary 14, $\gamma_P(G) \leqslant 2$ or $\gamma_P(\overline{G}) \leqslant 2$. Assume without loss of generality that $\gamma_P(G) \leqslant 2$. Applying Theorem 18 to $\overline{G}$, $\gamma_P(\overline{G}) \leqslant \lfloor \frac{n}{4} \rfloor$ for $n \geqslant 24$ and $\gamma_P(\overline{G}) \leqslant \lfloor \frac{n}{4} \rfloor + 1$ for $n \leqslant 23$. $\square$

Theorem 34. *Suppose G is a 3-regular graph of order $n \geqslant 6$ such that no component is $K_{3,3}$. Then $\gamma_P(G) \leqslant \lfloor \frac{n}{4} \rfloor$, $\gamma_P(\overline{G}) \leqslant 2$, and $\gamma_P(G) + \gamma_P(\overline{G}) \leqslant \lfloor \frac{n}{4} \rfloor + 2$, and all these inequalities are tight for arbitrarily large n.*

Proof. Suppose first that G is connected. Then $\gamma_P(G) \leqslant \lfloor \frac{n}{4} \rfloor$ by Theorem 20 (since $G \neq K_{3,3}$), so it suffices to show $\gamma_P(\overline{G}) \leqslant 2$. Since $G \neq K_4$ and G is 3-regular, $\operatorname{diam}(G) \geqslant 2$. Since $\operatorname{diam}(G) \geqslant 3$ implies $\gamma_P(\overline{G}) \leqslant 2$ by Theorem 8, we assume $\operatorname{diam}(G) = 2$. For any vertex v, there are at most 10 vertices at distance 0, 1, or 2 from v (v, its 3 neighbors, and two additional neighbors of each of the neighbors of v), so $n \leqslant 10$. An examination of 3-regular graphs with $6 \leqslant n \leqslant 10$ (see, for example, [18, p. 127]) shows the only such graphs of diameter 2 are the five graphs shown in Figure 6 (named as in [18]): C3 $= K_{3,3}$, C2, C5, C7, and C27 (the Petersen graph). It is straightforward to verify that $\gamma_P(\overline{G}) = 1$ for $G \in \{\text{C2, C7}\}$ and $\gamma_P(\overline{G}) = 2$ for $G \in \{\text{C5, C27}\}$. This completes the proof for the case in which G is connected.

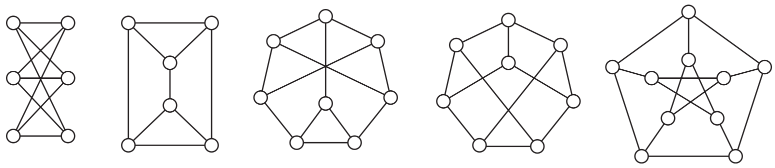

Figure 6: The five cubic graphs of diameter 2: C3 = $K_{3,3}$, C2, C5, C7, and C27 = the Petersen graph.

Now assume G has components $G_1, \dots, G_s$ with $s \geqslant 2$. Then $\gamma_P(\overline{G}) \leqslant 2$ by Theorem 8. Since $G_i \neq K_{3,3}$,

$$\gamma_P(G) = \sum_{i=1}^{s} \gamma_P(G_i) \leqslant \sum_{i=1}^{s} \left\lfloor \frac{n_i}{4} \right\rfloor \leqslant \left\lfloor \frac{\sum_{i=1}^{s} n_i}{4} \right\rfloor = \left\lfloor \frac{n}{4} \right\rfloor.$$

The graphs N_r attain the bound by Theorem 20 and Lemma 21. □

4 Nordhaus-Gaddum product bounds for power domination

As with the sum, the tight product lower bound for the power domination number for all graphs G remains unchanged even with the additional requirement that both G and $\overline{G}$ be connected (using the path). In Section 3, we achieved a tight sum upper bound for such graphs. However, since this was achieved with $\gamma_P(\overline{G}) = 1$ for both G and $\overline{G}$ connected, and with $\gamma_P(\overline{G}) = 2$ when each component of both G and $\overline{G}$ has order at least 3, there are few immediate implications for products (see Section 5 for further discussion of connections between sum and product bounds).

Question 35. Does there exist a graph G of order n such that all components of G and $\overline{G}$ have order at least 3 and $\gamma_P(G) \cdot \gamma_P(\overline{G}) > 2 \left\lfloor \frac{n}{3} \right\rfloor$?

Remark 36. If the answer to Question 35 is negative, then the graphs $G = rK_3$ with $r \geqslant 2$ show $2 \left\lfloor \frac{n}{3} \right\rfloor$ is a tight upper bound for the product, because $\gamma_P(G) = \frac{n}{3}$ and $\gamma_P(\overline{G}) = 2$.

Remark 37. If the answer to Question 26 is positive, then such graphs show $2 \left\lfloor \frac{n}{3} \right\rfloor$ can be attained for arbitrarily large n for the product with both G and $\overline{G}$ connected.

We can improve the product bound in certain special cases. The next result follows from Corollary 14 and Theorem 15.

Corollary 38. *Let G be a graph of order n such that every component of G and $\overline{G}$ has order at least 3. Then $\gamma_P(G) \cdot \gamma_P(\overline{G}) \leqslant 2 \left\lfloor \frac{n}{3} \right\rfloor$ if at least one of the following is true:*

1. $\operatorname{diam}(G) \geqslant 3$ *or* $\operatorname{diam}(\overline{G}) \geqslant 3$.

2. G *or* $\overline{G}$ *is planar.*

3. $\kappa(G) \leqslant 3$ *or* $\kappa(\overline{G}) \leqslant 3$.

4. G or $\overline{G}$ is not super-λ.

The next two results are product analogs of Theorems 32 and 33. The proofs, which are analogous, are omitted.

Theorem 39. *Let G be a graph of order n that has n_1 isolated vertices and n_2 copies of K_2 as components such that n_1 and n_2 are not both zero. Then*

$$\gamma_P(G) \cdot \gamma_P(\overline{G}) \leqslant \frac{n}{3} + \frac{2n_1}{3} + \frac{n_2}{3}.$$

Theorem 40. *Suppose G is a graph of order n with $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 2$, and one of the following is true:*

1. *G or $\overline{G}$ is planar.*
2. *$\kappa(G) \leqslant 3$ or $\kappa(\overline{G}) \leqslant 3$.*
3. *G or $\overline{G}$ is not super-λ.*

If $n \geqslant 24$, then $\gamma_P(G) \cdot \gamma_P(\overline{G}) \leqslant 2\left\lfloor \frac{n}{4} \right\rfloor$, and $\gamma_P(G) \cdot \gamma_P(\overline{G}) \leqslant 2\left\lfloor \frac{n}{4} \right\rfloor + 2$ for $n \leqslant 23$.

The next result follows immediately from Theorem 34.

Corollary 41. *Suppose G is a 3-regular graph of order $n \geqslant 6$ with no $K_{3,3}$ component. Then $\gamma_P(G) \cdot \gamma_P(\overline{G}) \leqslant 2\left\lfloor \frac{n}{4} \right\rfloor$, and this bound is attained for arbitrarily large n.*

Proposition 42. *Let G be a tree on $n \geqslant 4$ vertices. If G is not $K_{1,3}$ or $K_{1,4}$, then*

$$\gamma_P(G) \cdot \gamma_P(\overline{G}) \leqslant \left\lfloor \frac{n}{3} \right\rfloor$$

and this bound is attained for arbitrarily large n.

Proof. Note first that since G is connected, $\gamma_P(G) \leqslant \left\lfloor \frac{n}{3} \right\rfloor$ by Theorem 15. If a tree is not a star, then its complement is also connected, and by Proposition 6, $\gamma_P(\overline{G}) = 1$. For a star graph $K_{1,n-1}$, we have $\gamma_P(K_{1,n-1}) \cdot \gamma_P(\overline{K_{1,n-1}}) = 2$, which is less than or equal to $\frac{n}{3}$ when $n \geqslant 6$. The bound is attained for arbitrarily large n because if G is constructed from any tree T by adding two leaves to each vertex of T, then $\gamma_P(G) = \frac{n}{3}$. □

5 Summary and discussion

Table 1 summarizes what is known about Nordhaus-Gaddum sum bounds for power domination number, domination number, and zero forcing number.

Both the sum and product upper and lower bounds for the domination number were determined by Jaeger and Payan in 1972 (see Theorem 2), and analogous bounds for power domination are immediate corollaries. Since then, there have been numerous improvements to the sum upper bound for domination number under various conditions on

Table 1: Summary of tight bounds for $\zeta(G)+\zeta(\overline{G})$ for $\zeta = \gamma_P, \gamma, \mathrm{Z}$

ζ & restrictions	lower	upper
γ_P	2	$n+1$
γ_P & all components of both G and $\overline{G}$ of order $\geqslant 3$ & $n \geqslant 21$	2	$\left\lfloor \frac{n}{3} \right\rfloor + 2$
γ_P & both G and $\overline{G}$ connected & $n \geqslant 25$	2	$\left\lceil \frac{n}{3} \right\rceil + 1$
γ & $n \geqslant 2$	3	$n+1$
γ & both G and $\overline{G}$ connected	3	$\left\lfloor \frac{n}{2} \right\rfloor + 2$
Z & $n \geqslant 2$	$n-2$	$2n-1$
Z & both connected	$n-2$	$2n-o(n)$

G and $\overline{G}$. Examples of such conditions include requiring every component of both G and $\overline{G}$ to have order at least 2 or requiring both to be connected or requiring both to have minimum degree at least 7. In Section 3 we established better upper bounds for the power domination number in the cases where both G and $\overline{G}$ are connected or both have every component of order at least 3.

By contrast, results on products are very sparse for both domination number and power domination number. Historically, the Nordhaus-Gaddum sum upper bound has often been determined first, and then used to obtain the product upper bound, as in the case of Nordhaus and Gaddum's original results [16] (see Theorem 1). In order to use this technique of getting a tight product bound from a tight sum bound, one needs the sum upper bound to be optimized with approximately equal values or the sum lower bound to be optimized on extreme values. The sum lower bound for the domination number is optimized at the extreme values, and therefore the tight lower bound for the sum yields a tight lower bound for the product. However, all available evidence suggests that, for both the domination number and the power domination number, the sum upper bound is optimized only at extreme values. For example, the sum upper bound of $n+1$ over all graphs is attained only by the values 1 and n for both the domination and power domination numbers. Thus, for the domination number and the power domination number, the Nordhaus-Gaddum product upper bound presents challenges.

Further evidence indicating that the sum bound is optimized only on extreme values comes from random graphs. And it is also interesting to consider the "average'" behavior, or expected value, of the sum and product of Z, γ, and γ_P using the Erdős Rényi random graph $G(n, \frac{1}{2})$ (whose complement is also a random graph with edge probability $\frac{1}{2}$). Let $G = G(n, \frac{1}{2})$.

Let $\mathrm{tw}(H)$ denote the tree-width of a graph H. It is well known that $\mathrm{tw}(H) \leqslant \mathrm{Z}(H) \leqslant n$ for all graphs of order n. Since $\mathrm{tw}(G) = n - o(n)$ [17], $\mathrm{Z}(G) = n - o(n)$. Thus $\mathrm{Z}(G) + \mathrm{Z}(\overline{G}) = 2n - o(n)$ and $\mathrm{Z}(G) \cdot \mathrm{Z}(\overline{G}) = n^2 - o(n^2)$, and this establishes the upper bound listed in Table 1 for connected graphs G and $\overline{G}$, because G and $\overline{G}$ are both connected with probability approaching 1 as $n \to \infty$.

For any fixed $\epsilon > 0$, $\gamma(G) \leqslant (1+\epsilon)\log_2 n$ with probability going to 1 as $n \to \infty$ [15, 20]. Since $\gamma_P(H) \leqslant \gamma(H)$ for all graphs H, $\gamma_P(G) + \gamma_P(\overline{G}) \leqslant 2(1+\epsilon)\log_2 n << \lceil \frac{n}{3} \rceil + 1$ as $n \to \infty$ for $G = G(n, \frac{1}{2})$.

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