Power Domination in Cylinders, Tori, and Generalized Petersen Graphs

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A set S of vertices is defined to be a power dominating set (PDS) of a graph G if every vertex and every edge in G can be monitored by the set S according to a set of rules for power system monitoring. The minimum cardinality of a PDS of G is its power domination number. In this article, we find upper bounds for the power domination number of some families of Cartesian products of graphs: the cylinders $P_n \square C_m$ for integers $n \geq 2$, $m \geq 3$, and the tori $C_n \square C_m$ for integers $n, m \geq 3$. We apply similar techniques to present upper bounds for the power domination number of generalized Petersen graphs $P(m, k)$. We prove those upper bounds provide the exact values of the power domination numbers if the integers m, n, and k satisfy some given relations. **NETWORKS, Vol. 000(00), 000–000 2010**

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1. INTRODUCTION

Electric power companies need to monitor the state of their networks continually. The state of an electrical power network is defined by a set of variables: the voltage magnitude at loads and the machine phase angle at generators [1]. One method of monitoring these variables is to place phase measurement units (PMUs) at selected locations in the system. Because of the high cost of a PMU, it is important to minimize the number of PMUs needed to monitor the entire system.

This problem was first studied in terms of graphs by Haynes et al. in 2002 [7]. Indeed, an electrical power network can be modeled by a graph where the vertices represent the electrical nodes and the edges are associated with the transmission lines joining two electrical nodes. In this model, the power domination problem in graphs consists of finding a minimal set of vertices from which the entire graph can be observed according to certain rules. In terms of the physical network, those vertices will provide the locations where the PMUs should be placed in order to monitor the entire graph at a minimal cost.

A PMU measures the voltage and phase angle at the vertex where it is located, but also at other vertices or edges, according to the following propagation rules:

1. Any vertex that is incident to an observed edge is observed.
2. Any edge joining two observed vertices is observed.
3. If a vertex is incident to a total of k edges, $k > 1$, and if $k - 1$ of these edges are observed, then all k of these edges are observed.

Note that we followed the rules as presented in Ref. [7]. In Ref. [3] the authors present the propagation rules in a different way, that ultimately, as observed in Ref. [4], is equivalent to ours.

Algorithmically, given a graph $G = (V, E)$ and set of vertices $P \subset V$, we are going to construct a set of vertices C that can be observed from P and a set of edges F that are observed by P [4].

1. Initialize $C = P$ and $F = \{e \in E : e$ is incident to a vertex in $P\}$.
2. Add to C any vertex in $V - C$, which is incident to an edge in F.
3. Add to F any edge e in $E - F$, which satisfies one of the following conditions:
   (a) both end-vertices of e are in C.
   (b) e is incident to a vertex v of degree greater than one, for which all the other edges incident to v are already in F.
4. If steps 2 and 3 fail to locate any new edges or vertices for inclusion, stop. Otherwise, go to step 2.

The final state of the set C determines if P is a power dominating set (PDS). The power domination problem for

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a given graph G consists of finding a minimal PDS for G. The cardinality of a minimal PDS in G is called the power domination number of G, and it is denoted $\gamma_P(G)$. A PDS of G with cardinality $\gamma_P(G)$ will be referred to as a γ_P-set.

Given a graph G and a positive integer k, the power domination decision problem consists of deciding if the graph has a PDS of size at most k. The power domination decision problem has been proven to be NP-complete [7], even when reduced to certain classes of graphs, such as bipartite graphs or chordal graphs [7], or even split graphs [9], a subclass of chordal graphs. However, Liao and Lee [9] presented a linear time algorithm for solving this problem on interval graphs, if the interval ordering of the graph is provided. If the interval order is not given, they provided a quasi linear algorithm and proved that it is asymptotically optimal. Other efficient algorithms have been presented for trees [8] and more generally, for graphs with bounded treewidth [5]. On block graphs [12] and claw-free graphs [13], there are upper bounds given for the power domination number.

2. DEFINITIONS AND NOTATION

A graph $G = (V, E)$ is a pair formed by a nonempty set of vertices $V = V(G)$ and a set of edges $E = E(G)$ that contains unordered pairs of vertices. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V : uv \in E\}$. The closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. Similarly, for any set of vertices S, $N(S) = \cup_{v\in S}N(v)$ and $N[S] = \cup_{v\in S}N[v]$. The degree of a vertex v, denoted $\deg(v)$, is the cardinality of the set $N(v)$. The maximum and minimum degrees of a graph are defined as $\Delta(G) = \max\{\deg(v) : v \in V\}$ and $\delta(G) = \min\{\deg(v) : v \in V\}$, respectively. Let us denote the path of order n as P_n with vertex set $V(P_n) = \{x_i : 0 \le i \le n-1\}$ and the cycle of order m as C_m with vertex set $V(C_m) = \{y_j : 0 \le j \le m-1\}$.

The graphs we study in this article are cylinders and tori. Both, cylinders and tori, can be defined by means of the Cartesian product of graphs, so we recall its definition. Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the Cartesian product of G_1 and G_2 is the graph $G_1\square G_2$ whose set of vertices is the Cartesian product of the sets V_1 and V_2, so $V(G_1\square G_2) = V_1 \times V_2$. Two vertices of $G_1\square G_2$, say (v_1, v_2) and (u_1, u_2), are adjacent if and only if $v_1 = u_1$ and v_2 is adjacent with u_2 in G_2, or alternatively, if v_1 is adjacent with u_1 in G_1 and $v_2 = u_2$. For particular pairs of graphs G_1 and G_2, the Cartesian product provides well-known families of graphs. For example, the $n \times m$ grid can be obtained as the Cartesian product $P_n\square P_m$, where P_n and P_m, respectively, denote the paths with n and m vertices. Analogously, the $n \times m$ cylinder is the Cartesian product $P_n\square C_m$, where P_n denotes, as above, the path on n vertices, and C_m stands for the cycle on m vertices. The $n \times m$ torus is the Cartesian product of two cycles, say C_n and C_m, respectively, having n and m vertices.

In Ref. [4], the authors found a closed formula for $\gamma_P(G)$ when $G = P_n\square P_m$. We recall their main result next.

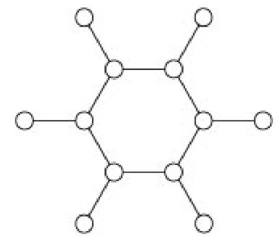

FIG. 1. The spiked cycle SC_6.

Theorem ([4]). *If G is an $n \times m$ grid graph, $m \ge n \ge 1$ then*

$$\gamma_P(G) = \begin{cases} \lceil \frac{n+1}{4} \rceil & \text{if } m \equiv 4 \text{ mod } 8 \\ \lceil \frac{n}{4} \rceil & \text{otherwise} \end{cases}.$$

∎

In Ref. [3], the number $\gamma_P(G)$ was calculated for graphs G arising from the strong, direct and lexicographic product of paths.

In this article, we partially extend the result in Ref. [4] for $P_n\square P_m$ to the cylinders $P_n\square C_m$ for integers $n \ge 2$, $m \ge 3$, and the tori $C_n\square C_m$ for integers $n, m \ge 3$.

We refer the reader to Refs. [2] and [6] for basic graph theory terminology and concepts.

3. CYLINDERS

In this section, we give the power domination number for some subgraphs of cylinders. We then proceed by giving an upper bound for the power domination number of cylinders using the found power domination numbers. Before we begin, we first present the following lemma that was shown in Ref. [7] and then recall the definition of a spiked cycle.

Lemma ([7]). *If G is a graph with $\Delta(G) \ge 3$, then G contains a $\gamma_P(G)$-set such that* $\deg(v) \ge 3$ *for each vertex in the $\gamma_P(G)$-set.*

The spiked cycle with $2m$ vertices, denoted as SC_m (Fig. 1), is the graph with vertex set $\{v_0, \dots, v_{m-1}\} \cup \{u_0, \dots, u_{m-1}\}$ and edges v_iv_{i+1} and v_iu_i, for all $i = 0, \dots, m-1$ with the subscript addition modulo m. Note that the vertices $\{v_0, \dots, v_{m-1}\}$ are cyclically connected by the edges v_iv_{i+1} while the edges v_iu_i add a spoke on each vertex of the cycle.

Lemma 3.1. *For any positive integer m, the power domination number of SC_m is $\gamma_P(SC_m) = \lceil \frac{m}{3} \rceil$.*

Proof. ($\le$) Let $S = \{v_i : i \equiv 0 \bmod 3\}$. Then, S is a PDS for SC_m with $|S| = \lceil \frac{m}{3} \rceil$. Thus, $\gamma_P[SC_m] \le \lceil \frac{m}{3} \rceil$.

($\ge$) Let S be a γ_P-set for SC_m. As $\Delta[SC_m] = 3$, we can assume that $\deg(v) = 3$, for all $v \in S$. In particular, every vertex in S lies on the cycle of SC_m. Now, suppose that $\gamma_P[SC_m] < \lceil \frac{m}{3} \rceil$. Then, there is a vertex $v_i \in SC_m$ such that

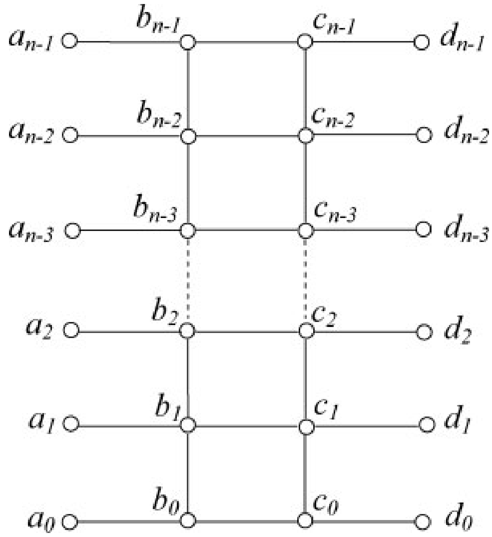

FIG. 2. The spiked band SB_n.

$d(v_i, N[S]) = 1$. So, there is a vertex $v_i \notin N[S]$. This vertex will be observed only if at least one edge incident to it is observed. The edge v_iu_i cannot be observed unless v_iv_{i+1} and $v_{i-1}v_i$ are observed. The edge v_iv_{i+1} is not observed since $u_{i+1}v_{i+1}$ is not observed (and vice versa). The edge v_iv_{i-1} is not observed since $u_{i-1}v_{i-1}$ is not observed (and vice versa). Hence, S is not a PDS. This completes the proof. ∎

Notice that for the cylinder $P_n \square C_m$, if the set of vertices $\{(x_0, y_j) : 0 \le j \le m-1\}$ is observed, then $P_n \square C_m$ is observed. To see this, consider the vertex $v = (x_0, y_k)$ and note that $\deg(v) = 3$. As the set of vertices $\{(x_0, y_j) : 0 \le j \le m-1\}$ is observed, then (x_0, y_{k-1}), (x_0, y_k), and (x_0, y_{k+1}) are observed, thus also (x_1, y_k) is observed. It follows that the set of vertices $\{(x_1, y_j) : 0 \le j \le m-1\}$ is observed. We can repeat this argument until the cylinder $P_n \square C_m$ is observed.

Lemma 3.2. *For any positive integers n and m, $\gamma_P(P_n \square C_m) \le \gamma_P(SC_m)$.*

Proof. Define the set S by $S = \{(x_0, y_i) : i \equiv 0 \bmod 3\}$ for $i \le m-1$. Then, every vertex in the set $\{(x_0, y_j) : 0 \le j \le m-1\}$ is observed and hence, the set S observes the graph $P_n \square C_m$. ∎

Given a positive integer n, the spiked band SB_n (Fig. 2) has a vertex set that can be partitioned into the union of four sets as it follows: $\{a_0, \ldots, a_{n-1}\} \cup \{b_0, \ldots, b_{n-1}\} \cup \{c_0, \ldots, c_{n-1}\} \cup \{d_0, \ldots, d_{n-1}\}$. Its edges are of two different types. The horizontal edges are a_ib_i, b_ic_i, and c_id_i for all $i = 0, \ldots, n-1$. The vertical edges are b_ib_{i+1} and c_ic_{i+1} for all $i = 0, 1, \ldots, n-1$.

In the illustrations throughout the remainder of this article, transparent vertices will represent vertices that are not observed, black vertices will represent observed vertices, and circled black vertices will represent the vertices in a PDS of the graph.

Lemma 3.3. *For any positive integer n, the power domination number of SB_n is $\gamma_P(\mathrm{SB}_n) = \lceil \frac{n+1}{2} \rceil$.*

Proof. $(\le)$ Define the set S as $S = \{b_i : i \equiv 0 \bmod 4\} \cup \{c_i : i \equiv 2 \bmod 4\}$ if n is even and $S = \{b_i : i \equiv 0 \bmod 4\} \cup \{c_i : i \equiv 2 \bmod 4\} \cup \{b_{n-1}\}$ if n is odd for $i \le n-1$ and $j \le n-1$. Then, S is a PDS with the desired cardinality.

$(\ge)$ Note that $\gamma_P(\mathrm{SB}_2) > 1$. Also, the construction given above for a PDS of SB_2 gives $\gamma_P(\mathrm{SB}_2) \le 2$. Thus, $\gamma_P(\mathrm{SB}_2) = 2$ and so we assume that $n \ge 3$. Suppose $\gamma_P(\mathrm{SB}_n) < \lceil \frac{n+1}{2} \rceil$. As $\Delta(\mathrm{SB}_n) = 4$, there is a γ_P-set, say S, such that $\deg(v) \ge 3$ for all $v \in S$. That is, each vertex in S is either b_i or c_i for some $i = 0, 1, \ldots, n-1$. As we assume that $\gamma_P(\mathrm{SB}_n) < \lceil \frac{n+1}{2} \rceil$ and each vertex in S is either b_i or c_i for some $i = 0, 1, \ldots, n-1$, then there exists some b_k or some c_k such that $b_k \notin N[S]$ or $c_k \notin N[S]$. Assume without loss of generality that $b_k \notin N[S]$. Either $\deg(b_k) = 3$ or $\deg(b_k) = 4$.

CASE 1. $\deg(b_k) = 3$. Then, $k = 0$ or $k = n-1$. Assume $k = 0$. Then, $b_1, c_0 \notin S$ and so $a_1, d_0 \notin N[S]$. As $\deg(b_1) = 4$ $\deg(c_0) = 3$ and since at most two neighbors of b_1 are observed and at most one neighbor of c_0 is observed, propagation does not give that b_0 is observed. Hence, SB_n is not observed and so $\gamma_P(\mathrm{SB}_n) \ge \lceil \frac{n+1}{2} \rceil$ when $\deg(b_k) = 3$. The proof is similar for $k = n-1$.

CASE 2. $\deg(b_k) = 4$. As $b_k \notin N[S]$, then $b_{k-1}, b_{k+1}, c_k \notin S$ and hence $a_{k-1}, a_{k+1}, d_k \notin N[S]$. Note that $\deg(b_{k-1}) = \deg(b_{k+1}) = \deg(c_k) = 4$. Then, for each of b_{k-1}, b_{k+1}, c_k, there exists two neighbors that are not observed and so propagation does not observe b_k. It follows that SB_n is not observed and so $\gamma_P(\mathrm{SB}_n) \ge \lceil \frac{n+1}{2} \rceil$ when $\deg(b_k) = 4$.

The desired lower bound follows from Case 1 and Case 2 and equality follows for $\gamma_P(\mathrm{SB}_n)$. ∎

Figure 3 illustrates Case 1 and Case 2 of the above proof.

Lemma 3.4. *For any positive integers n and m, $\gamma_P(P_n \square C_m) \le \gamma_P(\mathrm{SB}_n)$.*

Proof. Define the set S as $S = \{(x_0, y_i) : i \equiv 0 \bmod 4\} \cup \{(x_0, y_i) : i \equiv 2 \bmod 4\}$ if n is even and define S as $S = \{(x_0, y_i) : i \equiv 0 \bmod 4\} \cup \{(x_0, y_i) : i \equiv 2 \bmod 4\} \cup \{(x_0, y_{n-1})\}$ if n is odd for $i \le n-1$ and $j \le n-1$. Then, S is a PDS with $|S| = \lceil \frac{n+1}{2} \rceil$. ∎

Consolidating Lemma 3.2 and Lemma 3.4 gives

Lemma 3.5. *For any positive integers n and m, the power dominating number of $P_n \square C_m$ is bounded above by $\gamma_P(P_n \square C_m) \le \min\{\gamma_P(\mathrm{SB}_n), \gamma_P(SC_m)\}$.* ∎

We now present two upper bounds for the power domination number of cylinders.

Lemma 3.6. *The power domination number for $P_n \square C_m$ is bounded above by $\gamma_P(P_n \square C_m) \le \lceil \frac{m+1}{4} \rceil$.*

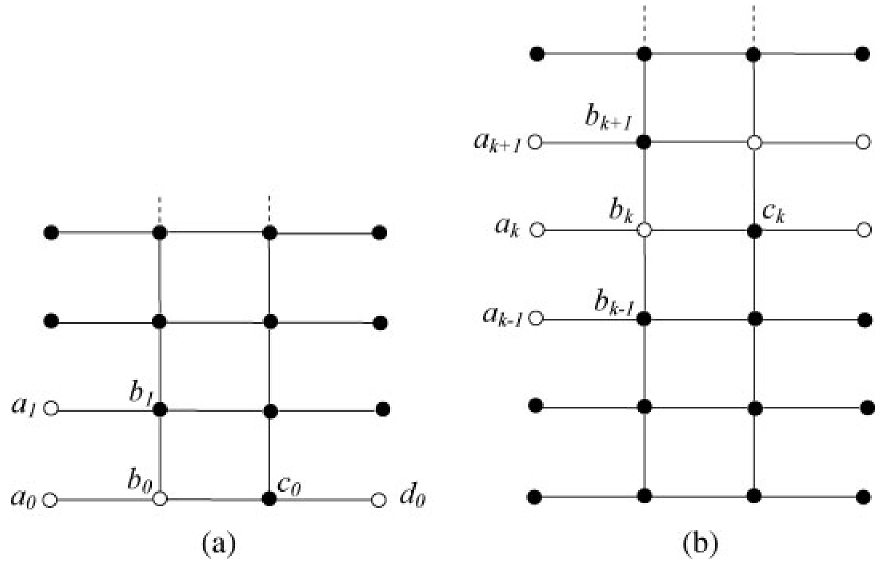

FIG. 3. Proof of Lemma 3.3. (a) Case 1: $\deg(b_0) = 3$. (b) Case 2: $\deg(b_k) = 4$.

Proof. Define the set S by $S = \{(x_k, y_0) : k \leq n - 1, k \equiv 0 \bmod 4\} \cup \{(x_l, y_1) : l \leq n - 1, l \equiv 2 \bmod 4\}$ for $k, l \leq \lceil \frac{m-1}{2} \rceil$. Then, S observes the set of vertices $\{(x_0, y_i) : 0 \leq i \leq m - 1\}$ and so S is a PDS with $|S| = \lceil \frac{m+1}{4} \rceil$. ∎

Figure 4 illustrates the PDS described in the above proof for the graph $P_n \Box C_7$ for $n \geq 4$.

As mentioned earlier, but in a more general manner, when the set of vertices $\{(x_0, y_j) : 0 \leq j \leq 7\}$ is observed, then $P_n \Box C_7$ is observed. For this reason, we only leave a lower bound on n, namely $n \geq 4$. The following upper bound was shown in the proof for Lemma 3.4.

Lemma 3.7. *The power domination number for $G = P_n \Box C_m$ is bounded above by $\gamma_P(G) \leq \lceil \frac{n+1}{2} \rceil$.* ∎

Consolidating Lemma 3.7 and Lemma 3.8 gives the following result:

Theorem 3.8. *The power domination number for the graph $P_n \Box C_m$ is $\gamma_P(P_n \Box C_m) \leq \min\{\lceil \frac{m+1}{4} \rceil, \lceil \frac{n+1}{2} \rceil\}$.* ∎

Note that Theorem 3.9 implies that $\gamma_P(P_n \Box C_m) = 1$ if $m \leq 3$ or $n = 1$, so the bound is attained for the infinite family of graphs $P_n \Box C_3$. We now use Theorem 3.9 to determine the power domination numbers for some other classes of cylinders. First realize that for $n \geq 2$ and $m \geq 4$, the power domination number of $P_n \Box C_m$ is bounded below by $\gamma_P(P_n \Box C_m) \geq 2$.

Corollary 3.9. *The power domination number for the graph $P_n \Box C_m$ is $\gamma_P(P_n \Box C_m) = 2$ if $n = 2, 3$ and $m \geq 4$, or if $n \geq 2$ and $4 \leq m \leq 7$.* ∎

4. TORI

As the torus $C_n \Box C_m$ is isomorphic to $C_m \Box C_n$, we assume $n \leq m$ for tori.

Lemma 4.1. *In the graph $C_n \Box C_m$, if the set of vertices $\{(x_i, y_j) : 0 \leq j \leq n - 1\} \cup \{(x_{i+1}, y_j) : 0 \leq j \leq n - 1\}$ is observed for some $0 \leq i \leq m - 2$, then the torus $C_n \Box C_m$ is observed.*

Proof. Let the set of vertices $\{(x_i, y_j) : 0 \leq j \leq n - 1\} \cup \{(x_{i+1}, y_j) : 0 \leq j \leq n - 1\}$ be observed for some $0 \leq i \leq m - 2$. As $\deg(x_{i+1}, y_j) = 4$ for all j and the set of vertices $N[(x_{i+1}, y_j)] \setminus \{(x_{i+2}, y_j)\}$ is observed for all j, it follows that the vertex (x_{i+2}, y_j) is observed for all $0 \leq j \leq n - 1$. Once (x_k, y_j) is observed for all j and for some $i + 2 \leq k$, we can follow a similar argument to show (x_{k+1}, y_j) is observed. In particular, we can repeat the argument until the torus $C_n \Box C_m$ is observed. ∎

Theorem 4.2. *The power domination number for the graph $G = C_n \Box C_m$, $n \leq m$, is*

$$\gamma_P(G) \leq \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 2 \bmod 4 \\ \lceil \frac{n+1}{2} \rceil & \text{otherwise} \end{cases}.$$

Proof. To show the upper bound above, we construct a PDS S that observes the vertices $\{(x_i, y_j) : 0 \leq j \leq n - 1\} \cup \{(x_{i+1}, y_j) : 0 \leq j \leq n - 1\}$. Define S as $S = \{(x_i, y_k) : k \equiv 1 \bmod 4\} \cup \{(x_{i+1}, y_l) : l \equiv 3 \bmod 4\}$ for $k, l \leq n - 1$ and if $n \equiv 2 \bmod 4$, then define S as $S = \{(x_i, y_k) : k \equiv 1 \bmod 4\} \cup \{(x_{i+1}, y_l) : l \equiv 3 \bmod 4\} \cup \{(x_{i+1}, y_1)\}$. Then, S is a PDS with $|S| = \lceil \frac{n+1}{2} \rceil$ if $n \equiv 2 \bmod 4$ and $|S| = \lceil \frac{n}{2} \rceil$ otherwise. ∎

It is easy to show that $\gamma_P(C_3 \Box C_m) > 1$ and as a consequence of Theorem 4.2, $\gamma_P(C_3 \Box P_m) = 2$. Therefore, there is an infinite family of graphs for which these bounds provide the exact value of the power domination number.

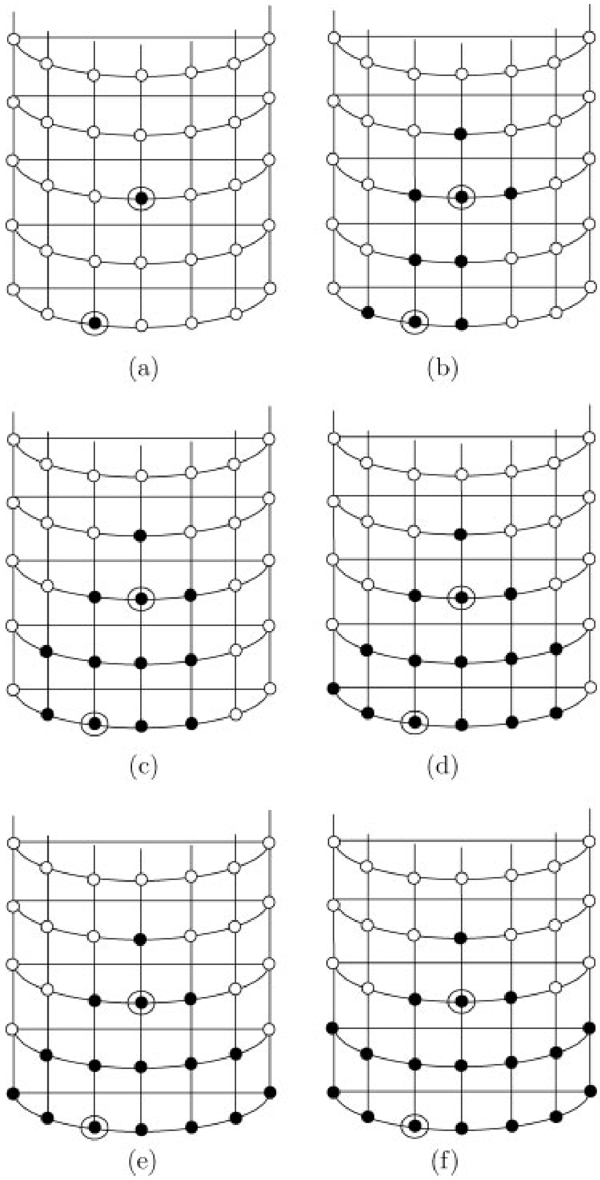

FIG. 4. Proof of Lemma 3.6. (a) Initial set S. (b) Closed neighborhood of S. (c) First propagation. (d) Second propagation. (e) (x_0, y_i) is observed. (f) Propagation continues.

5. GENERALIZED PETERSEN GRAPHS

We begin by recalling the definition of a generalized Petersen graph. For $m \geq 3$, $m > k \geq 1$, and $\gcd(m, k) = 1$, the generalized Petersen graph $P(m, k)$ is the graph with vertex set $\{v_0, v_1, \ldots, v_{m-1}\} \cup \{w_0, w_1, \ldots, w_{m-1}\}$ and edges $\{v_i w_i\}, \{v_i v_{i+1}\}, \{w_i w_{i+k}\}$ for every $i = 0, 1, \ldots, m-1$, where the subscript sum is taken modulo m. The vertices $\{v_0, \ldots, v_{m-1}\}$ will be referred to as outer vertices and the vertices $\{w_0, \ldots, w_{m-1}\}$ will be referred to as inner vertices. The $v_i w_i$ edges will be referred to as spokes. Figure 5 shows an example of a generalized Petersen graph.

We first present an upper bound on the power domination number for the generalized Petersen graph $P(m, k)$.

Lemma 5.1. *The power domination number for the generalized Petersen graph $P(m, k)$ satisfies $\gamma_P(P(m, k)) \leq k$.*

Proof. Define the set S by $S = \{w_0, w_1, \ldots, w_{k-1}\}$. We claim that S is a PDS. To see this, first notice that $N[S] = \{w_{m-k}, \ldots, w_0, \ldots, w_{2k-1}\} \cup \{v_0, v_1, \ldots, v_{k-1}\}$. Now, $N(v_{k-1}) = \{v_{k-2}, w_{k-1}, v_k\}$ and since w_{k-2}, v_{k-1}, and w_{k-1} are already dominated, then v_k is also dominated. Repeating the same argument successively with $v_k, v_{k+1}, \ldots, v_{2k-2}$ we can conclude that the vertices $v_{k+1}, \ldots, v_{2k-1}$ are also dominated. Analogously, if the process starts with v_0 instead of v_{k-1} and continues with the vertices $v_m, v_{m-1}, \ldots, v_{m-k-1}$, we conclude that $v_{k+1}, \ldots, v_{m-k}$ are also dominated. Thus, the set of dominated vertices so far is $\{w_{m-k}, \ldots, w_0, \ldots, w_{2k-1}\} \cup \{v_{m-k}, \ldots, v_0, \ldots, v_{2k-1}\}$. If $m \leq 3k$, then $P(m, k)$ is already dominated. Otherwise, we can increase the size of the dominated set by applying the following argument: $N(v_{2k-1}) = \{v_{2k-2}, w_{2k-1}, v_{2k}\}$ but w_{2k-1}, v_{2k-2} and v_{2k-1} are already dominated, so v_{2k} is also dominated. Besides, $N(w_k) = \{v_k, w_0, w_{2k}\}$ but w_k, v_k, and w_0 are already dominated, so w_{2k} is also dominated. Now, the set of dominated vertices is $\{w_{m-k}, \ldots, w_0, \ldots, w_{2k}\} \cup \{v_{m-k}, \ldots, v_0, \ldots, v_{2k}\}$.

Repeating the previous process we can add a vertex of type w_i and a vertex of type v_i in each step until the set of dominated vertices becomes $\{w_{m-k}, \ldots, w_0, \ldots, w_{m-k-1}\} \cup \{v_{m-k}, \ldots, v_0, \ldots, v_{m-k-1}\}$. That is, until all vertices are dominated. ∎

Figure 6 illustrates the idea used in the proof of Lemma 5.1.

Corollary 5.2. *Let m be an odd integer, $m \geq 5$. The power domination number for the generalized Petersen graph $P(m, 2)$ is $\gamma_P(P(m, 2)) = 2$.*

Proof. Since $m \geq 5$ then $\gamma_P(P(m, 2)) \geq 2$, because any set containing only one vertex can only dominate its closed neighborhood. Lemma 5.1 gives $\gamma_P(P(m, 2)) \leq 2$, and hence $\gamma_P(P(m, 2)) = 2$. ∎

Notice that from the definition of $P(m, k)$, $P(m, k) \cong P(m, m-k)$. Therefore, by Lemma 5.1, we know that

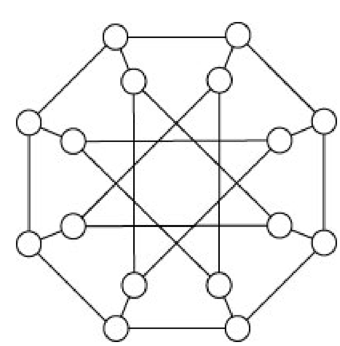

FIG. 5. The generalized Petersen graph $P(8, 3)$.

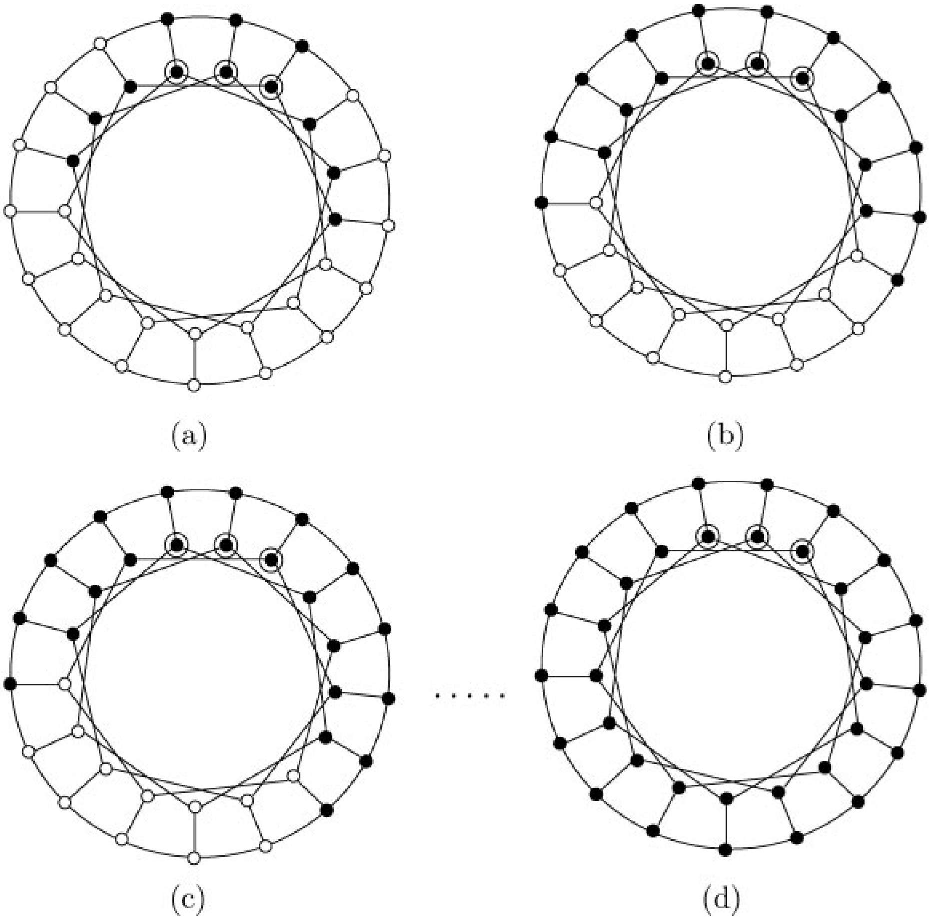

FIG. 6. Proof of Lemma 5.1. (a) Closed neighborhood of S. (b) Propagation in the outer cycle. (c) Additional propagations, if needed. (d) $P(17, 3)$ is observed.

$\gamma_P(P(m,k)) \leq \min\{k, m-k\}$. There are further characterizations of isomorphic generalized Petersen graphs that allow us to improve the bounds provided by Lemma 5.1. The following result, first proven by Watkins [11] and then by Steimle and Staton [10], will allow us to extend the results in Corollary 5.2 and improve the bounds provided in Lemma 5.1 for the power domination number of some generalized Petersen graphs.

Theorem ([10, 11]). *Let $m > 3$ and $\gcd(m,k) = 1$, $\gcd(m,l) = 1$, and $kl \equiv 1 \bmod m$. Then, $P(m,k) \cong P(m,l)$.* ∎

As a consequence, for any odd integer m, $m \geq 5$, and any integer l such that $\gcd(m,l) = 1$ and $2l \equiv 1 \bmod m$, the generalized Petersen graphs $P(m, 2)$ and $P(m,l)$ are isomorphic, so using Corollary 5.2 we can establish the following result.

Corollary 5.3. *Let m be an odd integer, $m \geq 5$. For any integer l such that $\gcd(m,l) = 1$ and $2l \equiv 1 \bmod m$, then the power domination number for the generalized Petersen graph $P(m,l)$ is $\gamma_P(P(m,l)) = 2$.* ∎

Furthermore, as a consequence of Corollary 5.3, we know that $\gamma_P(P(2p-1, p)) = 2$ for any prime integer $p \geq 3$.

Besides, as a consequence of the result in Refs. [10] and [11], we can also provide an improved version of Lemma 5.1. Take for example, the graphs $P(17, 9)$ and $P(17, 6)$. As a consequence of Lemma 5.1, we know that $\gamma_P(P(17, 9)) \leq 9$ and $\gamma_P(P(17, 6)) \leq 6$. Using that $P(17, 9) \cong P(17, 8)$ together with Lemma 5.1, we obtain $\gamma_P(P(17, 9)) \leq 8$. However, with some investigation, one will find that $\gamma_P(P(17, 9)) = 2$ and $\gamma_P(P(17, 6)) \leq 3$. The reason is that, as a consequence of the result from Refs. [10] and [11] previously mentioned, $P(17, 9) \cong P(17, 2)$ and $P(17, 6) \cong P(17, 3)$. These inequalities, together with Lemma 5.1, give $\gamma_P(P(17, 9)) = \gamma_P(P(17, 2)) = 2$ and $\gamma_P(P(17, 6)) = \gamma_P(P(17, 3)) \leq 3$.

Lemma 5.4. *The power domination number for the generalized Petersen graph $P(m,k)$ is bounded above by $\gamma_P(P(m,k)) \leq l'$ where $l' = \min\{l : P(m,k) \cong P(m,l)\}$.*

We omit the proof of Lemma 5.4 because it is essentially the same as the proof for Lemma 5.1.

6. OPEN PROBLEM

Let $(\Gamma, \cdot)$ be a group and let X be a generating set of Γ. The Cayley graph $C(\Gamma; X)$ is the graph with vertex set Γ and edges yz if $y \cdot x = z$ for some $x \in X$. Generalized Petersen graphs are a particular case of Cayley graphs. Whether the techniques used in this article can be generalized to any Cayley graph is an interesting open problem.

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