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Component Connectivity of Generalized Petersen Graphs

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Let G be a simple non-complete graph of order n. The r-component edge connectivity of G denoted as $\lambda_r(G)$ is the minimum number of edges that must be removed from G in order to obtain a graph with (at least) r connected components. The concept of r-component edge connectivity generalizes that of edge connectivity by taking into account the number of components of the resulting graph. In this paper we establish bounds of the r component edge connectivity of an important family of interconnection network models, the generalized Petersen graphs. Our investigation into this problem led us to solve another open problem: determining the girth of each generalized Petersen graph.
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1. Introduction

Let G be a simple non-complete graph of order n. An r-component cut of G is a set of vertices whose removal yields a graph with at least r connected components. The r-component connectivity of G denoted as $\kappa_r(G)$ is the minimum cardinality of an r-component cut. That is, $\kappa_r(G)$ is the minimum number of vertices that must be removed from G in order to obtain a graph with at least r connected components. Therefore, $\kappa_2(G) = \kappa(G)$, the connectivity of G. Notice that $\kappa_r(G)$ is defined as the minimum number of vertices that must be removed from G in order to obtain a graph with *at least* r connected components, since for some graphs G it is not possible to obtain a graph with *exactly* r connected components by removing vertices. This concept was originally introduced by Sampathkumar [2] and has been recently studied for hypercubes by Hsu et al. in [3].

An r-component edge cut of G is a set of edges whose removal yields a graph with r connected components. The r-component edge connectivity of G denoted as $\lambda_r(G)$ is the minimum cardinality of an r-component edge cut. That is, $\lambda_r(G)$ is the minimum number of edges that must be removed from G in order to obtain a graph with r connected components. Notice that $\lambda_2(G) = \lambda(G)$, the edge connectivity of G. Also, notice that the minimum number of edges that must be removed from G in order to obtain a graph with *at least* r connected components coincides with the minimum number of edges that must be removed from G in order to obtain a graph with *exactly* r connected components. The concept of r-component edge connectivity was introduced by Sampathkumar [2] who called it general line connectivity.

The following are some important inequalities regarding r-component connectivity and r-component edge connectivity of an arbitrary simple non-complete graph G of order n. First, $\lambda_r(G) \leq \lambda_{r+1}(G)$ for $r = 2, \dots, n-1$. Also $\kappa_r(G) \leq \kappa_{r+1}(G)$ for $r = 2, \dots, n-1$.

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Moreover, $\kappa_r(G) \leq \lambda_r(G)$ for $r = 2, \ldots, n-1$. Proofs of these inequalities are found in [2].

In this paper we study $\kappa_r(G)$ and $\lambda_r(G)$ when G is a generalized Petersen Graph. Given integers $n \geq 3$ and $k \geq 1$, the generalized Petersen graph $GP(n,k)$ has $2n$ vertices labeled $u_0, u_1, \ldots u_{n-1}, v_0, v_1, \ldots v_{n-1}$. The vertices labeled $u_0, u_1, \ldots u_{n-1}$ are called outer vertices while those labeled $v_0, v_1, \ldots v_{n-1}$ are inner vertices. There are three types of edges in $GP(n,k)$: i) edges connecting outer vertices in the form $u_i u_{(i+1) \bmod n}$ for $i = 0, 1, \ldots n-1$; ii) edges connecting inner vertices in the form $v_i v_{(i+k) \bmod n}$ for $i = 0, 1, \ldots n-1$; and iii) spokes, or edges connecting an outer vertex with an inner one, in the form $u_i v_i$ for $i = 0, 1, \ldots, n-1$ for $i = 0, 1, \ldots n-1$. Since k is only used within the modular arithmetic, k is restricted to $1 \leq k < n$ without loss of generality.

From the definition of generalized Petersen graphs it follows that $GP(n,k) \cong GP(n, n-k)$. Moreover, in some cases it is possible to find integers l, other than k and $n-k$, such that $GP(n,k) \cong GP(n,l)$. The following result from [5] specifies the isomorphism class of $GP(n,k)$.

Theorem 1.1. *Let n be an integer, $n \geq 5$. Let k and l be integers relatively prime to n satisfying $2 \leq k, l \leq n-2$. If $GP(n,k) \cong GP(n,l)$, then either $l = \pm k \bmod n$ or $kl = \pm 1 \bmod n$.*

From this point use the smallest value of k in the isomorphism class of $GP(n,k)$. As a consequence, we can always restrict k to values between 1 and $\left\lfloor \frac{n}{2} \right\rfloor$. Furthermore, if n is even, then $\frac{n}{2}$ is an integer and is not relatively prime to n. Thus for minimal k, we can conclude $1 \leq k < \frac{n}{2}$.

In order to study the r-component edge connectivity of generalized Petersen graphs we use the girth of a graph. Let G be a simple graph with at least one cycle, then the girth of G, denoted as $g(G)$, is defined as the minimum among the lengths of all cycles in G. A shortest cycle is a cycle of minimum length. Some bounds for the girth of a generalized Petersen graph were presented in [4]. In this paper we establish the exact value of $g(G)$ for every generalized Petersen graph G.

We refer the reader to [1] for background concepts in graph theory not defined in this Introduction.

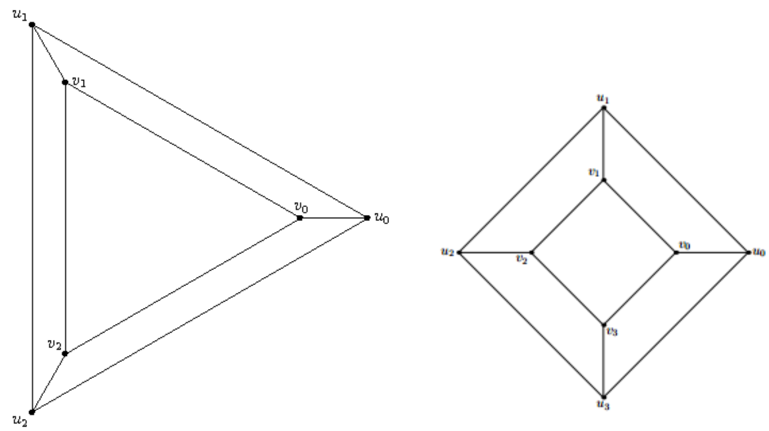
2. Girth

In this section we will establish the exact value of the girth of a generalized Petersen graph $GP(n,k)$ for any integers $n \geq 3$ and $k \geq 1$. We begin with two special cases when $n = 3$ and $n = 4$. First, note that for an arbitrary graph G, the trivial lower bound for the girth is $g(G) \geq 3$. The graph $GP(3,1) \cong GP(3,2)$, clearly has girth 3, as seen in Figure 1(a). We will prove later that this is the only graph to have girth 3. In the $n = 4$ case, $GP(4,1) \cong GP(4,3)$ has girth 4, as seen in Figure 1(b). Any cycle must contain an even number of spokes, and no two spokes are directly adjacent. If a cycle of $GP(4,1)$ has no spokes then it is either $\langle u_0, u_1, u_2, u_3, u_0 \rangle$ or $\langle v_0, v_1, v_2, v_3, v_0 \rangle$, in both cases of length 4. If a cycle contains two spokes, then if must contain at least two other edges or the spokes would be adjacent. Thus any such cycle had length greater than or equal to 4.

The following theorem is a result from [4] which provides an upper bound for the girth of $GP(n,k)$:

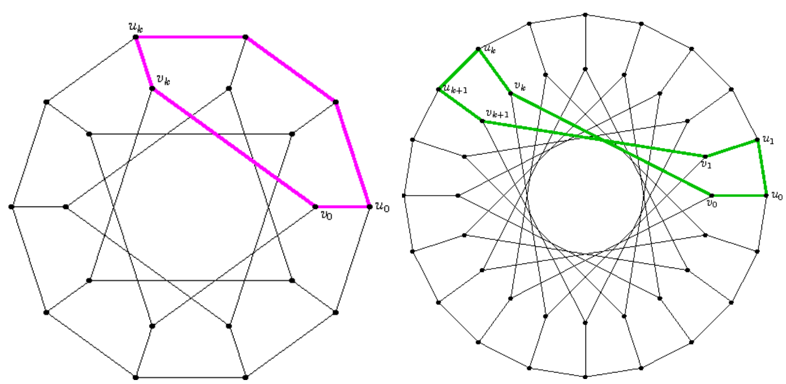
Theorem 2.1. *[4] For every $n \geq 5$, $1 < k < \frac{n}{2}$, and k relatively prime to n, the girth, $g(GP(n,k)) \leq \min\{8, k+3\}$.* □

In the study of cycles in generalized Petersen graphs we define different patterns.



(a) $GP(3, 1)$, with $g(G) = 3$ (b) $GP(4, 1)$, with $g(G) = 4$

Figure 1. The Generalized Petersen Graphs of small order.



(a) The wedge cycle in $GP(10, 3)$ (b) The figure eight cycle in $GP(10, 3)$

Figure 2. Two different common cycles: the wedge cycle and the figure eight cycle.

A wedge cycle, as seen in Figure 2(a), has the form $\langle u_i, u_{i+1}, \ldots, u_{i+k}, v_{i+k}, v_i, u_i\rangle$. It contains exactly one inner edge, two spokes and the outer edges connecting those spokes. A figure eight cycle, as seen in Figure 2(b), has the form $\langle u_i, u_{i+1}, v_{i+1}, v_{i+k+1}, u_{i+k+1}, u_{i+k}, v_{i+k}, v_i, u_i\rangle$. This is the shortest cycle that contains four spokes, and can be found in any generalized Petersen graph. The upperbound established in Theorem 2.1 is established because wedge cycles and figure eight cycles are found in every generalized Petersen Graph.

Lemma 2.2. *Let n and k be two relatively prime integers, $n \geq 5$, such that $k = \min\{l : GP(n, l) \cong GP(n, k)\}$. If $g(GP(n, k)) \neq 8$, there is a shortest cycle in $GP(n, k)$ that contains exactly two spokes. If $g(GP(n, k)) = 8$, there is a shortest cycle that contains exactly four spokes.*

Proof. Since spokes are the only edges joining outer vertices with inner ones, if a cycle contains spokes, it must contain an even number of them. Furthermore, since no two spokes are incident with the same vertex, for each spoke there must be at least one additional edge in the cycle. Therefore, any cycle with five or more spokes must have length greater than or equal to 10. But, since $g(GP(n, k)) \leq 8$ for integers $n \geq 5$ and $1 \leq k < \frac{n}{2}$, all cycles of minimum length have at most four spokes. Furthermore, the only case in which there is a cycle of minimum length with four spokes is when the girth is 8.

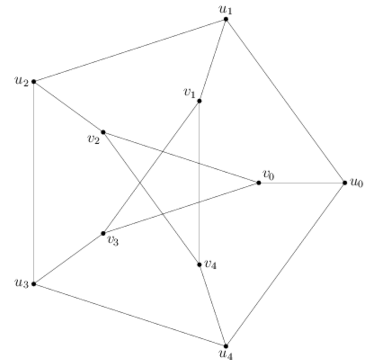


Figure 3. The Petersen Graph, $GP(5, 2)$

Notice that a cycle that contains no spokes must exclusively have outer vertices or inner vertices, but cannot have both. Besides, since n is relative prime to k, any cycle in $GP(n, k)$ which contains no spokes must be of length n. However, since we assume k to be the minimal in the isomorphism class, $k < \frac{n}{2}$. Since $3 < \frac{n}{2}$ for $n > 6$, we get that $k + 3 < \frac{n}{2} + \frac{n}{2} = n$. Since the girth of $GP(n, k)$ is less than or equal to $k + 3$ for all integers $n \geq 5$ and $1 \leq k < \frac{n}{2}$, it is impossible for a cycle of minimum length to have length n if $n > 6$. Still, since we assuming $n \geq 5$ we still need to analyze the cases $n = 5$ and $n = 6$. Calculating the cases, we conclude that only the graph $GP(5, 2)$ has a cycle of minimal length n that contains no spokes. Furthermore, note that $GP(5, 2)$ has cycles of minimal length 5 with two spokes, such as $\langle u_0, u_1, v_1, v_4, u_4, u_0\rangle$, see Figure 3.

In conclusion, in a graph $GP(n, k)$ where $n \geq 5$ and $1 \leq k < \frac{n}{2}$ every cycle of minimum length must have two or four spokes, except in the graph $GP(5, 2)$. In the case $GP(5, 2)$ there are cycles of minimal length which contain two spokes. Furthermore, the only case in which there is a cycle of minimum length with four spokes is when the girth is 8. □

Lemma 2.3. *Let n and k be two relatively prime integers, $n \geq 5$, such that $k = \min\{l : GP(n, l) \cong GP(n, k)\}$. Then, $g(GP(n, k)) \geq 4$.*

Proof. It suffices to show $g(GP(n, k)) \neq 3$. If there was a cycle of length three, from Lemma 2.2 it would contain exactly two spokes. Since no two spokes are incident with the same vertex, the length of a cycle with two spokes must be at least four, and this is a contradiction. □

Lemma 2.4. *Let n and k be two relatively prime integers, $n \geq 5$, such that $k = \min\{l : GP(n, l) \cong GP(n, k)\}$. Then, $g(GP(n, k)) = 4$ if and only if $k = 1$.*

Proof. Assume $g(GP(n, k)) = 4$. By Lemma 2.2 there exists a shortest cycle C that contains exactly two spokes. However, since the cycle has length four, the other two edges must both be incident with each spoke. Therefore the cycle is a wedge cycle with the form $\langle u_i, u_{i+1}, v_{i+1}, v_i, u_i\rangle$. However, since v_i is only adjacent to v_{i+k}, v_{i-k} and u_i, for v_i to be adjacent to v_{i+1} it means $k = 1$ or $k = n - 1$. Since k is assumed minimal, $k = 1$.

Conversely, if $k = 1$, then $GP(n, 1)$ has a cycle $\langle u_i, u_{i+1}, v_{i+1}, v_i, u_i\rangle$, and since by Lemma 2.3 there are no cycles of length three, then $g(GP(n, k)) = 4$. □

Lemma 2.5. *Let n and k be two relatively prime integers, $n \geq 5$, such that $k = \min\{l : GP(n, l) \cong GP(n, k)\}$. Then, $g(GP(n, k)) = 5$ if and only if $k = 2$.*

Proof. Assume $g(GP(n, k)) = 5$. By Lemma 2.2 there is a shortest cycle C that contains two spokes, say u_iv_i and u_jv_j for some integers i, j, $0 \leq i, j \leq n-1$. The remaining edges in C connect either two outer or two inner vertices. In the first case, C is a wedge cycle with $\langle u_i, v_i, v_{j=i+k}, u_{j=i\pm 2}, u_{i\pm 1}, u_i\rangle$, which clearly indicates that $k = 2$ by minimality of k. In the second case, C will include the path $\langle u_i, v_i, v_{i+k}, v_{j=i+2k}, u_{j=i\pm 1}, u_i\rangle$ and as a consequence, $i + 2k = i \pm 1 \bmod n$, so $2k = \pm 1 \bmod n$. However, this means that 2 and k are in the same isomorphism class, and the minimality of k implies $k = 2$.

Conversely, observe that by construction $GP(n, 2)$ contains a wedge cycle of length 5. Then $g(GP(n, 2)) \leq 5$. By Lemma 2.3 and Lemma 2.4 $GP(n, 2)$ does not contain cycles of length 3 or 4, so we conclude $g(GP(n, k)) = 5$. □

Lemma 2.6. *Let n and k be two relatively prime integers, $n \geq 5$, such that $k = \min\{l : GP(n, l) \cong GP(n, k)\}$. Then, $g(GP(n, k)) = 6$ if and only if $k = 3$ or $n = 2k + 2$.*

Proof. Assume $g(GP(n, k)) = 6$. By Lemma 2.2 there is a shortest cycle C that contains two spokes. Therefore, there are three cases for C depending on whether the remaining four edges connect inner vertices or outer vertices:

Case (i): There are three edges connecting outer vertices and one edge connecting inner vertices. In this case, C has the form of a wedge cycle. As a result $k = \pm 3 \bmod n$, but by the minimality of k, we conclude that $k = 3$.

Case (ii): There are two edges connecting outer vertices and two edges connecting inner vertices. In this case, C has the form $\langle u_i, v_i, v_{i+k}, v_{j=i+2k}, u_{j=i\pm 2}, u_{i\pm 1}, u_i\rangle$, where in $u_{j=i\pm 2}$ and $u_{i\pm 1}$ both are either additions or subtractions. However, this means that $2k = \pm 2 \bmod n$. Since $k < \frac{n}{2}$ implies that $n > 2k$. This implies that $2k$ cannot equal $2 \bmod n$. Therefore, the only possibility is that $n = 2k + 2$.

Case (iii): There are one edge connecting outer vertices and three edges connecting inner vertices. In this case C has the form $\langle u_i, v_i, v_{i+k}, v_{i+2k}, v_{j=i+3k}, u_{j=i\pm 1}, u_i\rangle$. From $i + 3k = i \pm 1 \bmod n$ we can conclude that $3k = \pm 1 \bmod n$. This means 3 and k are the in same isomorphism class, so by the minimality of k we must conclude that $k \leq 3$. However, if $k < 3$, then the girth is less than 6 which contradicts the hypothesis.

To prove the converse, we use the following two cases depending on k:

Case (i): If $k = 3$, then we can construct a wedge cycle of length 6, which means $g(GP(n, k)) \leq 6$. Furthermore, by Lemmas 2.3, 2.4 and 2.5 $GP(n, k)$ does not contain cycles of length 3, 4, or 5, so we conclude $g(GP(n, k)) = 6$.

Case (ii): If $n = 2k + 2$, then $GP(n, k)$ contains a cycle of length 6 in the form $\langle u_i, v_i, v_{i+k}, v_{i+2k}, u_{i+2k=i+(n-2)}, u_{i+(n-1)}, u_i\rangle$, see Figure 4(c). This construction means that $g(GP(n, k)) \leq 6$. Again, by Lemmas 2.3, 2.4 and 2.5 we know $GP(n, k)$ can contain cycles of length 3, 4, or 5 only if $k = 1$ or $k = 2$. If $k = 1$, then $n = 4$ which contradicts the hypothesis $n \geq 5$. If $k = 2$, then $n = 6$ which contradicts the hypothesis that n and k are relatively prime. Therefore, $g(GP(n, k)) = 6$. □

Lemma 2.7. *Let n and k be two relatively prime integers, $n \geq 5$, such that $k = \min\{l : GP(n, l) \cong GP(n, k)\}$. Then, $g(GP(n, k)) = 7$ if and only if $k = 4$ or $n = 2k + 3$ for $k \geq 4$ or $n = 3k \pm 2$ for $k \geq 5$.*

Proof. Assume $g(GP(n, k)) = 7$. By Lemma 2.2 there is a shortest cycle C that contains exactly two spokes. We consider four cases depending on the remaining five edges in C:

Case (i): There are four edges connecting outer vertices and one edge connecting inner vertices. In this case C has the form of a wedge cycle. Therefore, $k = \pm 4$, so by the minimality of k we see that $k = 4$ in this case.

Case (ii): There are three edges connecting outer vertices and two edges connecting inner vertices. In this case C has the form $\langle u_i, v_i, v_{i+k}, v_{j=i+2k}, u_{j=i\pm 3}, u_{i\pm 2}, u_{i\pm 1}\rangle$. Therefore, $2k = \pm 3 \bmod n$. However, since $n > 2k$ by the minimality of k we know $2k \neq 3 \bmod n$. Therefore, $n = 2k + 3$.

Case (iii): There are two edges connecting outer vertices and three edges connecting inner vertices. In this case C has the form $\langle u_i, v_i, v_{i+k}, v_{i+2k}, v_{j=i+3k}, u_{j=i\pm 2}, u_{i\pm 1}, u_i\rangle$. Therefore, $3k = \pm 2 \bmod n$. Since $n > 2k$ we know that $3k < 2n$, and thus $n = 3k \pm 2$.

Case (iv): There is one edge connecting outer vertices and four edges connecting inner vertices. In this case C has the form $\langle u_i, v_i, v_{i+k}, v_{i+2k}, v_{i+3k}, v_{j=i+4k} u_{j=i\pm 1}, u_i\rangle$. This means that $4k = \pm 1 \bmod n$. However, this means 4 and k are in the same isomorphism class, so $k \leq 4$ by the minimality of k. But, $k \notin \{1, 2, 3\}$ because of Lemmas 2.3, 2.4, 2.5, and 2.6.

To prove the converse, we use the following three cases depending on k:

Case (i): If $k = 4$ we know that $GP(n, 4)$ the wedge cycle has length 7, so we may conclude $g(GP(n, k)) \leq 7$. Furthermore, by Lemmas 2.3, 2.4, and 2.5 $GP(n, 4)$ has no cycles of length 3, 4 or 5. In addition, by Lemma 2.6 $GP(n, 4)$ can have a cycle of length 6 only if $n = 2k + 2$ However, if $k = 4$ then $n = 2k + 2 = 10$ and since 4 and 10 are not relatively prime, that case cannot happen under the hypothesis. Thus, we conclude that $g(GP(n, k)) = 7$ in this case.

Case (ii): If $n = 2k + 3$ for $k \geq 4$, then $GP(n, k)$ has a cycle of length 7 of the form $\langle u_i, v_i, v_{i+k}, v_{j=i+2k}, u_{j=i-3}, u_{i-2}, u_{i-1}, u_i\rangle$ (see Figure 4(d)), so we may conclude $g(GP(n, k)) \leq 7$. Furthermore, because $k \geq 4$ and $n \neq 2k + 2$ by Lemmas 2.3, 2.4, 2.5, and 2.6 there are no cycles of length 3, 4, 5, or 6 . Thus, we conclude that $g(GP(n, k)) = 7$ in this case.

Case (iii): If $n = 3k \pm 2$ for $k \geq 5$, then $GP(n, k)$ has a cycle of length 7 of the form $\langle u_i, v_i, v_{i+k}, v_{i+2k}, v_{j=i+3k}, u_{j=i\pm 2}, u_{i\pm 1}, u_i\rangle$ (see Figures 4(e) and 4(f)), so we may conclude $g(GP(n, k)) \leq 7$. Since $k \geq 5$, $GP(n, k)$ contains no cycles of length 3, 4 or 5. In addition, if $n = 2k + 2$ and $n = 3k + 2$, then the only solution for k is $k = 0$, which contradicts the condition that $k \geq 5$. If $n = 2k + 2$ and $n = 3k - 2$, then the solution of the equation is $k = 4$, which also contradicts that $k \geq 5$. Therefore, it is not possible for $GP(n, k)$ to have a cycle of length 6, and we conclude that $g(GP(n, k)) = 7$ in this case. □

Theorem 2.8. *Let n and k be two relatively prime integers, $n \geq 5$, such that $k = \min\{l : GP(n, l) \cong GP(n, k)\}$.*

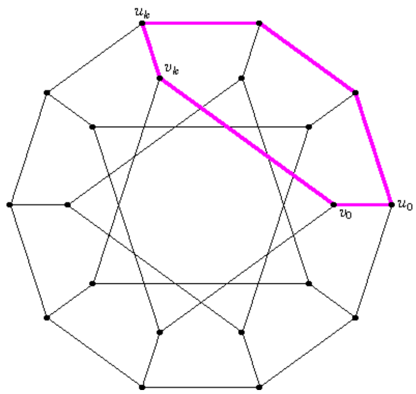
$$g(GP(n,k)) = \begin{cases} 4, \text{ if } k = 1 \\ 5, \text{ if } k = 2 \\ 6, \text{ if } n = 2k+2, \text{ or } k = 3 \\ 7, \text{ if } n = 2k+3 \text{ for } k \geq 4, \text{ or } n = 3k \pm 2 \text{ for } k \geq 5, \text{ or } k = 4 \\ 8, \text{ otherwise} \end{cases}$$

Proof. By Theorem 2.1 we know $g(GP(n, k)) \leq 8$. By Lemmas 2.3, 2.4, 2.5, 2.6 and 2.7 we know exactly when the girth is 4, 5, 6, and 7. Therefore, in all additional cases the girth must be 8. □

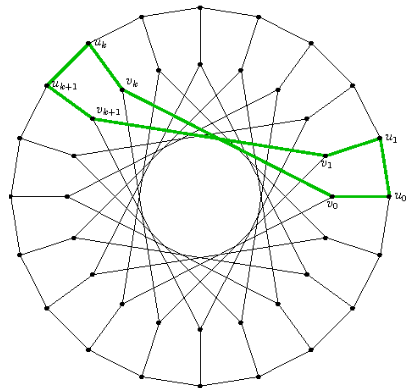
3. Edge component connectivity

In this section we provide lower and upper bounds for the r-component edge connectivity of a generalized Petersen graphs. The lower bounds, however, apply to any connected graph.

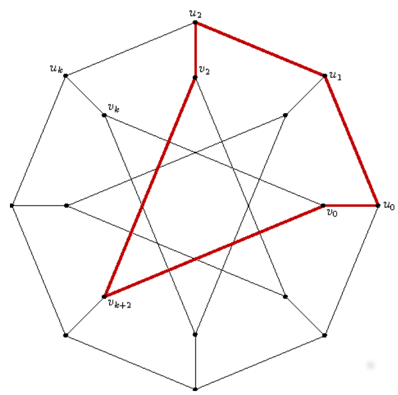
Theorem 3.1. *Let G be a connected graph with order n and edge connectivity $\lambda(G)$. For*



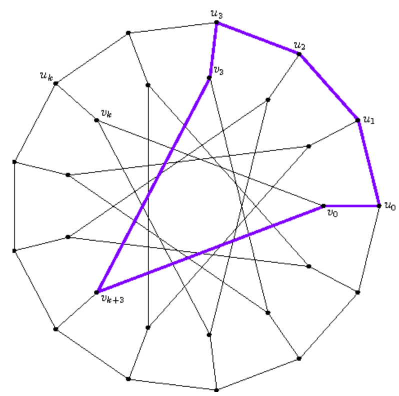
(a) The wedge cycle in $GP(10,3)$



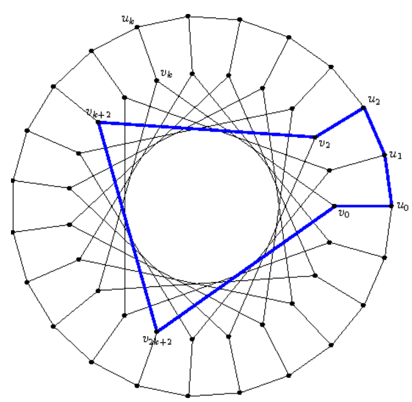
(b) A figure eight cycle in $GP(20,7)$



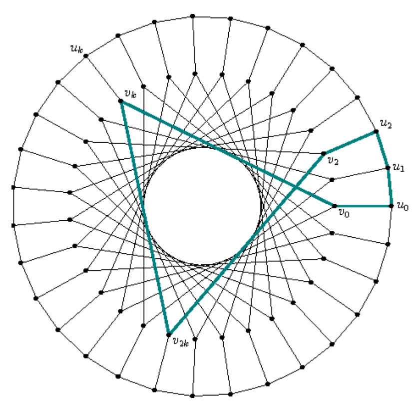
(c) A shortest cycle when $n = 2k+2$, as found in $GP(8,3)$



(d) A shortest cycle when $n = 2k+3$, as found in $GP(13,5)$



(e) A shortest cycle when $n = 3k+2$, as found in $GP(23,7)$



(f) A shortest cycle when $n = 3k-2$, as found in $GP(31,11)$

Figure 4. These figures show the six possibilities for a shortest cycle.

every integer $r = 2, \ldots, n$, let $\lambda_r(G)$ be the r-component edge connectivity of G. Then,

$$\lambda_r(G) \geq \left\lceil \frac{r\lambda(G)}{2} \right\rceil$$

Proof. Notice that from the definition of 2-component edge connectivity, $\lambda(G) = \lambda_2(G)$, so it is equivalent to prove $\lambda_r(G) \geq \left\lceil \frac{r\lambda_2(G)}{2} \right\rceil$. Let S_r be a minimal r-component edge cut. Then, $G - S_r$ contains exactly r components, say $C_1, C_2, \ldots, C_r$. Notice that for each $i = 2, \ldots, r$, the number of edges in S_r incident to vertices in C_i must be at least $\lambda_2(G)$

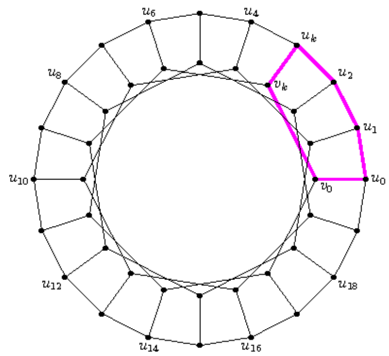


Figure 5. Beginning the (de)construction of $GP(20, 3)$. The wedge cycle, C_0, is bold and magenta. The cycle has minimum length

(otherwise G could have been disconnected by deleting less than $\lambda_2(G)$ edges, which is a contradiction). However, each edge in S_r will be incident with vertices in two different components, so we can conclude that $|S_r| = \lambda_r(G) \geq \left\lceil \frac{r\lambda_2(G)}{2} \right\rceil$. □

Since $\lambda(GP(n, k)) = 3$, we can derive the following corollary:

Corollary 3.2. *Let n and k be integers such that $n \geq 3$ and $k \geq 1$. Then, for every integer $r = 2, \ldots, n$,*

$$\lambda_r(GP(n, k)) \geq \left\lceil \frac{3r}{2} \right\rceil$$

Next we provide upper bounds for the r-component edge connectivity of generalized Petersen graphs. The bounds depend on the structure of the shortest cycles, so each of the six types of shortest cycles have their own theorem. All of the proofs are by construction and involve careful analysis of all the cycles of shortest length and how different copies of these cycles overlap. In each of the piecewise functions that follow, each piece of the function models a different way that the copies interact with one another.

Theorem 3.3. *Let n and k be two relatively prime integers, $n \geq 5$, such that $k = \min\{l : GP(n, l) \cong GP(n, k)\}$. If $g(GP(n, k)) = k + 3$, then for every integer $r = 2, \ldots, 2n$, $\lambda_r(GP(n, k)) \leq M_1(n, k, r)$, where*

$$M_1(n,k,r) = \begin{cases} 2r-1 & 2 \leq r \leq k+3 \\ 2r-1-\left\lfloor \frac{r-k-1}{3} \right\rfloor & k+4 \leq r \leq \min\{4k, 2n-4k\} \\ 2r-k-1-\left\lfloor \frac{r-4k-1}{2} \right\rfloor & 4k+1 \leq r \leq 2n-4k \\ 2r-n+3k-1-\left\lfloor \frac{2}{3}(r-2n+4k) \right\rfloor & 2n-4k+1 \leq r \leq 2n-k-1 \\ n+r & 2n-k \leq r \leq 2n \end{cases}$$

Proof. To prove this result, for each $r = 2, \ldots, 2n$ we will construct S_r, an r-component edge cut for $GP(n, k)$ with cardinality $M_1(n, k, r)$. Since $\lambda_r(GP(n, k))$ is defined as the minimum cardinality of an r-component edge cut of $GP(n, k)$, it follows immediately that $\lambda_r(GP(n, k)) \leq M_1(n, k, r)$.

Since $g(GP(n, k)) = k + 3$, for every $i = 0, \ldots, n-1$ the family of wedge cycles is the family of shortest cycles. Let C_i be the wedge cycle starting at u_i, $C_i = \langle u_i, u_{i+1}, \ldots, u_{i+k}, v_{i+k}, v_i, u_i \rangle$, where all indices are considered modulo n.

We will describe the construction process of each S_r using the cycles described above. Without loss of generality, let us assume we start the process with the cycle $C_0 =$

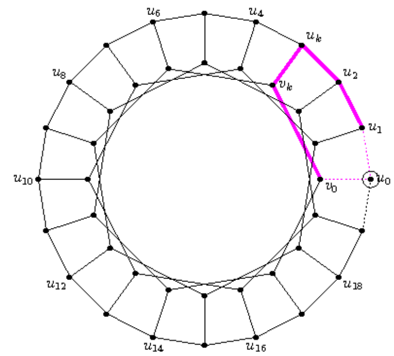


Figure 6. $GP(20,3) - S_2$ which has two components.

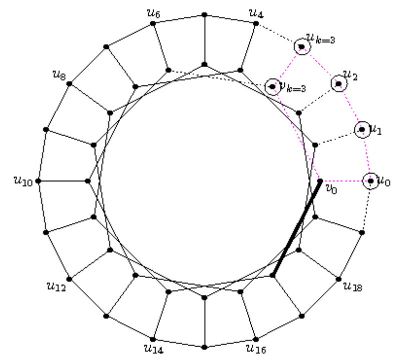


Figure 7. $GP(20,3) - S_{k+3}$ with $k+3=6$ components. The only vertex of C_0 in the large component is v_0, which is incident to a bridge that is shown in bold.

$\langle u_0, u_1, \ldots u_k, v_k, v_0, u_0 \rangle$, as seen in Figure 5. Notice that for every pair of integers n and k such that $n \geq 5$ and $k \geq 1$, $\lambda_2(GP(n,k)) = 3$. Then, the set of all edges incident to any given vertex form a minimal edge cut for $GP(n,k)$. In particular, removing the edges u_0u_{n-1}, u_0v_0 and u_0u_1 isolates u_0. Then, $S_2 = \{u_0u_{n-1}, u_0v_0, u_0u_1\}$ is a 2-component edge cut, as seen in Figure 6. We proceed constructing S_k by adding edges to S_{k-1} for $k = 3, \ldots, 2n$

For every integer $r = 3, \ldots, k+3$ we can build an r-component edge cut S_r by adding two edges to the $(r-1)$-component edge cut S_{r-1}. Indeed, isolating the remaining vertices of C_0 in the order specified in the description of C_0 above, guarantees that the removal of exactly two edges suffices to isolate the remaining vertices $u_1, u_2, \ldots, u_k, v_k$. Therefore, for each integer $r = 2, \ldots k+3$ the set S_r constructed has cardinality $3 + 2(r-2) = 2r-1$, so $M_1(n,k,r) = 2r-1$ if $2 \leq r \leq k+3$.

The vertex v_0 has not been isolated yet, as seen in Figure 7, and it is attached to the large component of the graph only by a bridge. Therefore, S_{k+4} can be obtained by adding one single edge to S_{k+3}, as in Figure 8. As a consequence, $M_1(n,k,k+4) = 2r-2$. Notice that $2r-2 = 2r-1-\left\lfloor\frac{k+4-k-1}{3}\right\rfloor = 2r-1-\left\lfloor\frac{r-k-1}{3}\right\rfloor$ when $r = k+4$.

Once all vertices of C_0 have been isolated, we proceed isolating the vertices of C_1. However, vertices $u_1, u_2, \ldots, u_k$ have already been isolated by the previous step. Therefore, only three vertices in C_1 remain incident to the large component: u_{k+1}, v_{k+1} and v_1, as seen in Figure 8. As we proceed in that order, two edges must be removed to isolate u_{k+1}, two edges must be removed to isolate v_{k+1} and one edge to isolate v_1. Thus, to create an additional component, we must increase the number of removed edges by two, except for every third component, where only one edge suffices.

Therefore, the function M_1 changes when $r \geq k+4$. After the first $k+4$ components, every third component will only be connected by a bridge. This means that $M_1(n,k,r) =$

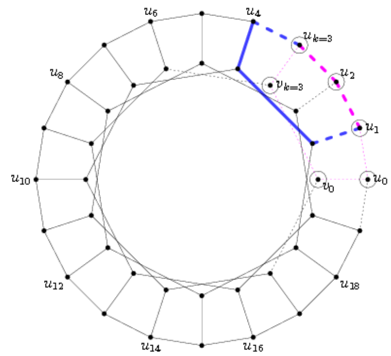


Figure 8. This illustrates the transition from the first line to the second line of M_1. $GP(20,3) - S_{k+4}$ contains $k+4 = 7$ components. Every vertex of C_0 is isolated, and C_1 is shown in bold.

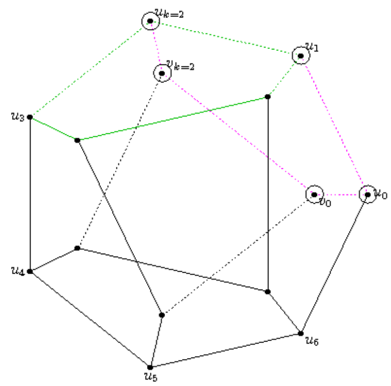


Figure 9. $GP(7,2)$ where $n - 2k < 2k$, i.e. $7 - 4 < 4$. Note that after we isolate v_3, the vertex v_5 is connected only by a bridge.

$2r-2-\lfloor \frac{r-k-4}{3} \rfloor = 2r-1-\lfloor \frac{r-k-1}{3} \rfloor$. Furthermore, if $n \geq 4k$, then each cycle $C_2, \ldots, C_{k-1}$ continues this pattern of having three vertices that are not isolated. When we isolate the vertices on all of these cycles $C_0, \ldots, C_{k-1}$, we have $k+3+3(k-1) = 4k$ isolated vertices, so $M_1(n,k,r) = 2r-1-\lfloor \frac{r-k-1}{3} \rfloor$ for $r \in [k+4, 4k]$ when $n \geq 4k$. However, when $n < 4k$ or equivalently, when $n - 2k < 2k$ (see Figure 9), we have that v_{n-2k} is a vertex on C_i for some $i < k$. When this occurs, the pattern changes and this piece of the function no longer applies. Therefore, the domain for this piece of the function is $r \in [k+4, \min\{4k, 2n-4k\}]$.

Next we continue, with C_k. Note, if $n < 4k$, then $[4k+1, 2n-4k]$ is the empty set, so we are only concerned with the case $n \geq 4k$. When we consider C_k, we observe that now $u_k, u_{k+1}, \ldots, u_{2k-1}$ and v_k have all been isolated by previous steps (see Figure 10). As such, the only two vertices connected to the large component are u_{2k} and v_{2k}.

Furthermore, as we continue around the graph, the cycles $C_{k+1}, C_{k+2}, \ldots, C_{n-3k}$ all have exactly two vertices not isolated by previous steps. As such, two edges will need to be removed to isolate each u_i and one edge to isolate each v_i, which means $M_1(n,k,r)$ changes every other component. Since this change begins with the $4k+1$ component, which has $2(4k+1)-k-1$ edges removed, we know $M_1(n,k,r) = 2r-k-1-\lfloor \frac{r-4k-1}{2} \rfloor$ for $r \in [4k+1, 2n-4k]$.

When we have exactly $r = 2n-4k$ components, this means the vertices u_0, u_1, $\ldots$, u_{n-2k}, v_0, v_1, $\ldots$, v_{n-2k-1} have been completely isolated and the vertices u_{n-2k+1}, u_{n-2k+2}, $\ldots$, u_{n-1}, v_{n-2k}, v_{n-2k+1}, $\ldots$, v_{n-1} are still connected (see Figure 11). At this point $M_1(n,k,r = 2n-4k) = 2r-k-1-\lfloor \frac{2n-4k-4k-1}{2} \rfloor = 2r-n+3k$.

When we isolate v_{n-2k} we remove the edge $v_{n-2k}v_{n-k}$, which makes $M_1(n,k,2n-4k+$

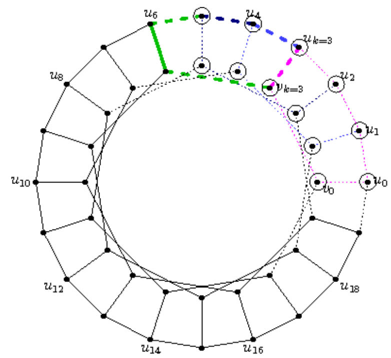


Figure 10. $GP(20, 3) - S_{4k+1}$ with $4k + 1 = 13$ components. The cycle C_k is shown in bold. Only two vertices of C_k are in the large connected component.

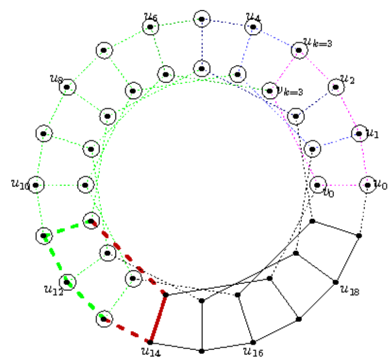


Figure 11. This illustrates the third line of M_1. $GP(20, 3) - S_{2n-4k+1}$ with $2n - 4k + 1 = 29$ components. The cycle C_{2n-3k} is in bold.

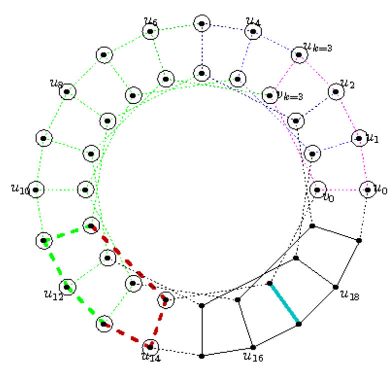


Figure 12. The transition from the third line to the fourth line of M_1. $GP(20, 3) - S_{2n-4k+3}$ with $2n-4k+3 = 31$ components. Note the bridge connecting $v_{n-k=17}$ to the large component, which is emphasized with bold and in cyan.

$1) = 2r - n + 3k - 1$. However, since $v_{n-k}v_0$ was removed in the first step, this means v_{n-k} can also be isolated by only removing one additional edge (namely, $u_{n-k}v_{n-k}$), see Figures 12 and 13. This pattern repeats for the next $k - 1$ cycles, where we isolate u_{n-2k+i} by removing two edges, we isolate v_{n-2k+i} by removing one edge, and we isolate v_{n-k+i} by removing one edge, where $i \in [0, k - 1]$. Thus, $M_1(n, k, r) = 2r - n + 3k - 1 - \left\lfloor \frac{2}{3}(r - 2n + 4k) \right\rfloor$, and increases by 1, two out of every three components for $r \in [2n - 4k + 1, 2n - k - 1]$.

After all of these vertices have been isolated, the graph has the form of $2n - k$ isolated vertices and a path of length k, as seen in Figure 14. Therefore, at this point we only need to remove one edge in order to isolate each vertex. As such, $M_1(n, k, r)$ will increase

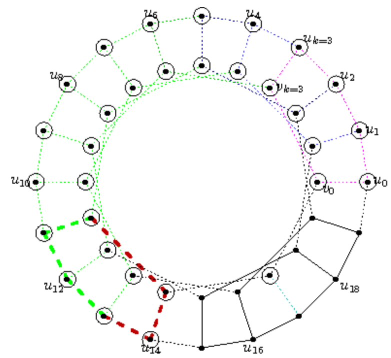


Figure 13. $GP(20,3) - S_{2n-4k+4}$ with $2n-4k+4 = 31$ components. The bridge connecting $v_{n-k=17}$ to the large component has been removed.

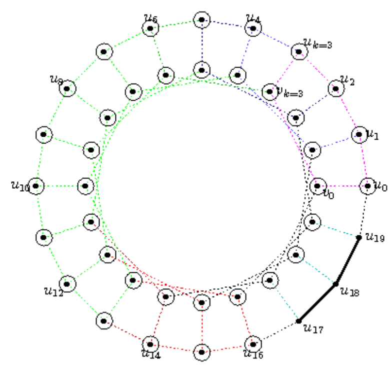


Figure 14. $GP(20,3)$ nearly disconnected. The only vertices that remain connected form a path of exactly k vertices, which is illustrated in bold.

by one every time the number of components increases by one. □

Theorem 3.4. *Let n and k be two relatively prime integers, $n \geq 5$, such that $k = \min\{l : GP(n,l) \cong GP(n,k)\}$. If $g(GP(n,k)) = 8$, then for every integer $r = 2,\ldots,2n$, we conclude $\lambda_r(GP(n,k)) \leq M_2(n,k,r)$, where*

$$M_2(n,k,r) = \begin{cases} 2r-1 & 2 \leq r \leq 8 \\ 2r-2-\left\lfloor \frac{r-9}{4} \right\rfloor & 9 \leq r \leq \min\{4k-6, 2n-4k\} \\ 2r-k+1 & r = 4k-5 < 2n-4k \\ 2r-k+1-\left\lfloor \frac{r-4k+5}{3} \right\rfloor & 4k-4 \leq r \leq 4k+1 \\ 2r-k-1-\left\lfloor \frac{r-4k-1}{2} \right\rfloor & 4k+2 \leq r \leq 2n-4k \\ 2r-n+3k-1-\left\lfloor \frac{2}{3}(r-2n+4k-1) \right\rfloor & 2n-4k+1 \leq r \leq 2n-k \\ n+r & 2n-k+1 \leq r \leq 2n \end{cases}$$

Proof. This proof is very similar to the previous one. Let $G = GP(n,k)$, where n and k satisfy all of the conditions in the hypothesis. The function $M_2(n,k,r)$ arises from creating an r-component edge cut, S_r, by isolating vertices along repeated copies of shortest cycles. Since $g(GP(n,k)) = 8$, the figure eight cycles, $C_i = \langle u_i, u_{i+1}, v_{i+1}, v_{i+k+1}, u_{i+k+1}, u_{i+k}, v_{i+k}, v_i, u_i \rangle$, are cycles of shortest length, see Figure 15. An important observation is that $C_i \cap C_{i+1}$ contains four vertices $\{u_{i+1}, v_{i+1}, v_{i+k+1}, u_{i+k+1}\}$.

As before, $S_2 = \{u_0u_1, u_0v_0, u_0u_{n-1}\}$. To construct, S_3 through S_8 we add the edges incident with each vertex of C_0, one at a time. As such $S_3 = S_2 \cup \{u_1u_2, u_1v_1\}$, $S_4 = S_3 \cup \{v_1v_{k+1}, v_1v_{n-k+1}\}$ and $S_8 = S_7 \cup \{v_kv_{2k}, v_0v_k\}$. Because of this $M_2(n,k,r) = |S_r| = 2r-1$

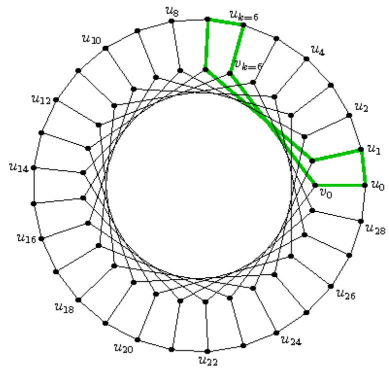


Figure 15. $GP(29,6)$ which has girth 8. The cycle C_0 is shown in bold and green.

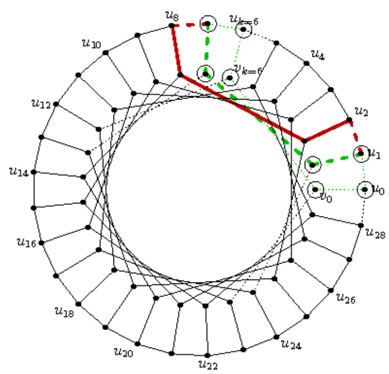


Figure 16. $GP(29,6) - S_9$ which has girth 8. The cycle C_1 is shown in bold and red.

for $2 \le r \le 8$.

In $G - S_8$, every vertex of C_0 is isolated, except the final vertex v_0. This vertex is connected by a bridge to $G - S_8$, so $S_9 = S_8 \cup \{v_0, v_{n-k}\}$. Since C_1 overlaps with C_0, in $G - S_9$ the vertices $u_1, v_1, v_{k+1}, u_{k+1}$ are isolated, see Figure 16. However, the vertices $u_2, v_2, v_{k+2}, u_{k+2}$ are still connected in the large component of $G - S_9$. Isolating the first three vertices requires an additional two edges in S_{10}, S_{11}, and S_{12}. However, isolating u_{k+2} will only require one additional edge, because this reaches the end of the cycle C_1. The pattern of adding one additional edge, then three instances of adding two additional edges repeats from S_9 until $S_{4(k-2)+1}$. This is because the pattern corresponds to the four vertices which remain connected in $C_1, C_2, \ldots, C_{k-3}$. As such, $M_2(n,k,r) = |S_r| = 2r - 2 - \left\lfloor \frac{r-9}{4} \right\rfloor$ for $9 \le r \le 4k - 7$.

$G - S_{4k-7}$ consists of the isolated vertices u_0, u_1, …, u_{k-3}, v_0, v_1, …, v_{k-3}, u_k, u_{k+1}, …, u_{2k-3}, v_k, v_{k+1}, …, v_{2k-3}, and one large connected component. Continuing along the pattern of isolating vertices in shortest cycles, we consider u_{k-2} the first vertex in C_{k-2}. Isolating u_{k-2} requires two edges, so $S_{4k-6} = S_{4k-7} \cup \{u_{k-2}u_{k-1}, u_{k-2}v_{k-2}\}$, which gives the second line of the piecewise function. However, now there is a slight change in the pattern. The vertex u_{k-1} is now connected by only a bridge because the edges $u_{k-1}u_k$ and $u_{k-2}u_{k-1}$ have previously been removed. Thus, $S_{4k-5} = S_{4k-6} \cup \{u_{k-1}v_{k-1}\}$, and $M_2(n,k,r = 4k-5) = 2r - 2 - \left\lfloor \frac{4k-5-9}{4} \right\rfloor + 1 = 2r - k + 1$.

However, now the vertices of C_{k-3} and C_{k-2} must be isolated. Since, three vertices of C_{k-3} are in the large connected component of $G - S_{4k-5}$, so $|S_{4k-4}| = 2r - k + 1$, $|S_{4k-3}| = 2r - k + 1$ and $|S_{4k-2}| = 2r - k + 2$. This pattern repeats with C_{k-2}, so we can summarize this as $M_2(n,k,r) = 2r - k + 1 - \left\lfloor \frac{r-k+5}{3} \right\rfloor$ for $4k - 4 \le r \le 4k + 1$.

In $G - S_{4k+1}$, six of the vertices of C_{k-1} are isolated, namely $\{u_{k-1}$, v_{k-1}, u_k, v_k,

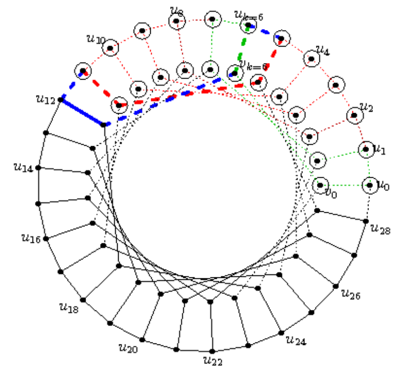


Figure 17. $GP(29,6) - S_{4k+1}$ which has girth 8. The cycle C_k is shown in bold and blue.

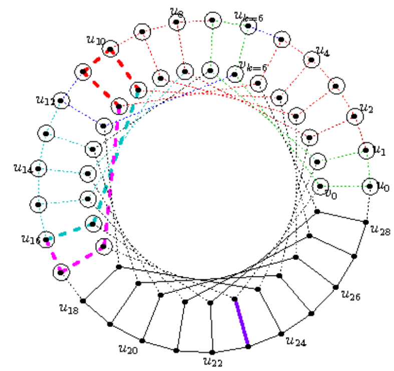


Figure 18. $GP(29,6) - S_{2n-4k+3}$ which has girth 8. The cycle $C_{10=n-3k-1}$ is shown in bold. In addition, the bridge $u_{23=n-k}v_{23=n-k}$ is shown in bold and in purple.

u_{2k-1}, $v_{2k-1}\}$. This leaves only u_{2k} and v_{2k} in the large connected component, see Figure 17. Two edges must be added to S_{4k+2} and one edge to S_{4k+3} in order to isolate these vertices. The pattern of each C_i containing two connected vertices continues through C_{n-3k-2}. These cycles correspond to edge cuts S_{4k+2} through S_{2n-4k}, where edge cuts with an odd index gain one additional edge while edge cuts with an even index gain two additional edges. As such $M_2(n,k,r) = 2r - k - \left\lfloor \frac{r-4k-1}{2} \right\rfloor$. The last vertex of C_{n-3k-2} is connected by a bridge, so $S_{2n-4k+1}$ gains one edge $v_{n-2k-2}v_{n-k-2}$.

The pattern changes slightly in C_{n-3k-1}. There are still two vertices of C_{n-3k-1} in the connected component of $G - S_{2n-4k+1}$. Thus, $S_{2n-4k+2}$ gains two edges $u_{n-2k}u_{n-2k+1}, u_{n-2k}v_{n-2k}$, and $S_{2n-4k+3}$ gains one edge $v_{n-2k}v_{n-k}$. However, the vertex v_{n-k} is now connected by a bridge (see Figure 18), because $v_{n-2k}v_{n-k}$ is in edge set $S_{2n-4k+3}$, and $v_{n-k}v_0$ is in edge set S_2. The only remaining edge adjacent to v_{n-k} is $u_{n-k}v_{n-k}$. Thus, $S_{2n-4k+4}$ gains only one additional edge. This pattern continue through C_{n-2k-2}, so $M_2(n,k,r) = 2r - n + 3k - \left\lfloor \frac{2}{3}(r - 2r + 4k - 1) \right\rfloor$ for $2n - 4k + 1 \leq r \leq 2n - k$.

At this point $G - S_{2n-k}$ is a path containing $k+1$ vertices, see Figure 19. Each edge removal results in a new component, so $M_2(n,k,r) = n + r$, until all $2n$ vertices are isolated, which means $3n$ edges have been removed. □

Theorem 3.5. *Let n and k be two relatively prime integers, $n \geq 5$, such that $k = \min\{l : GP(n,l) \cong GP(n,k)\}$. If $n = 2k+2$, then for every integer $r = 2, \ldots, 2n$, $\lambda_r(GP(n,k)) \leq M_3(n,k,r)$, where*

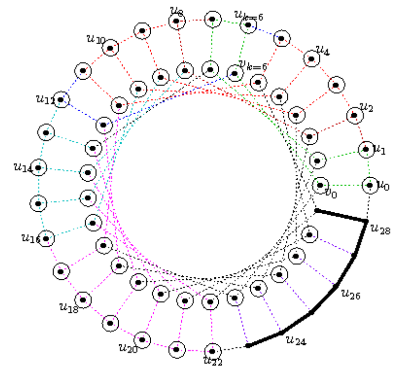


Figure 19. $GP(29,6) - S_{2n-k+1}$ which has girth 8. The path is shown in bold.

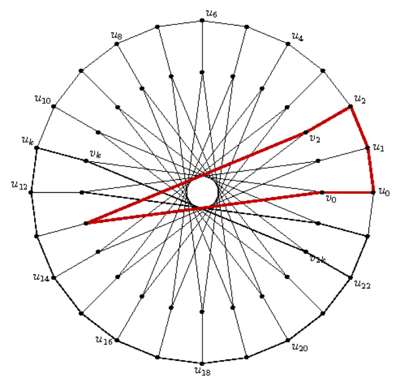


Figure 20. $GP(24,11)$ which has girth 6, and $24 = 2 \cdot 11 + 2$. One copy of the smallest cycle is bold and in red.

$$M_3(n,k,r) = \begin{cases} 2r-1 & 2 \le r \le 6 \\ 2r-2 & 7 \le r \le \min 10, 3k-1 \\ 2r-3-\left\lfloor \frac{r-11}{3} \right\rfloor & 11 \le r \le 3k-1 \\ 2r-k+1-\left\lfloor \frac{2}{3}(r-3k+2) \right\rfloor & 3k \le r \le 3k+3 \\ n+r & 2n-k = 3k+4 \le r \le 2n = 4k+4 \end{cases}$$

Proof. In the hypothesis of this theorem, the shortest cycles in $GP(n,k)$ are in the form $C_i = \langle u_i, u_{i+1}, u_{i+2}, v_{i+2}, v_{i+k+2}, v_{i+2k+2 \equiv i}, u_i \rangle$, as in Figure 20.

We begin by isolating each vertex in C_0. It requires and edge cut of $S_2 = 3$ edges to isolate u_0, and an additional two edges to isolate each of u_1, u_2, v_2 and $v_k + 2$. As such, $M_3(n,k,r) = 2r-1$ for $2 \le r \le 6$. The final vertex of C_0 is v_0 and requires only one additional edge for the edge cut S_7, so $M_3(n,k,7) = 2r-2$.

The next copy of the cycle is C_1. The vertices u_1 and u_2 are isolated in $G - S_7$, so there are only four vertices in the connected component. The isolation of the first three of these vertices (u_3, v_3, v_{k+3}) require the addition of two edges to the edge cut, so $M_3(n,k,r) = 2r-2$ for $7 \le r \le 10$. The final vertex, v_1 is attached by a bridge so $M_3(n,k,r) = 2r-3$.

Next, is C_2. In $G - S_{10}$, u_2, v_2 and u_3 are isolated vertices. v_{k+4}, v_4 and u_4 are connected. Therefore, S_{11} gains two additional edges, S_{12} gains two additional edges, and S_{13} requires one additional edge. This pattern continues with C_3 through C_{k-3}. When C_{k-3} is fully disconnected there are $3(k-4)+4+6+1 = 3k-1$ components. As such, $M_3(n,k,r) = 2r-3-\left\lfloor \frac{r-11}{3} \right\rfloor$ for $11 \le r \le 3k-1$.

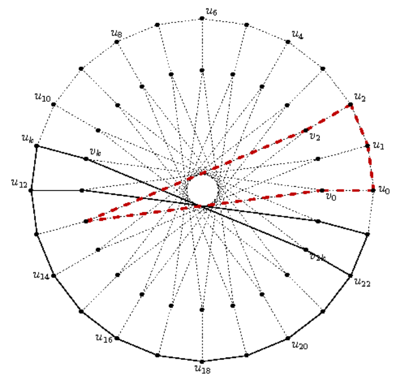


Figure 21. $GP(24,11) - S_{3k-1}$

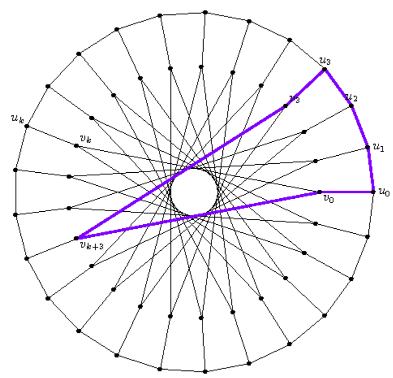


Figure 22. $GP(25,11)$ which has girth 7, and $25 = 2 \cdot 11 + 3$. One copy of the shortest cycle is bold and in purple.

In $G - S_{3k-1}$ the vertices $u_0, v_0, u_1, v_1, \ldots, u_{k-1}, v_{k-1}$ and $v_{k+2}, v_{k+3}, \ldots, v_{2k-1}$ have already been isolated, as seen in Figure 21. As we continue and isolate u_k, two additional edges must be removed, so $S_{3k} = S_{3k-1} \cup \{u_k u_{k+1}, u_k v_k\}$. In $G - S_{3k}$, the vertex v_k is connected by a bridge, so $S_{3k+1} = S_{3k} \cup \{v_k v_{2k}\}$. In $G - S_{3k+1}$, the vertex v_{2k} is also connected by a bridge, so $S_{3k+2} = S_{3k+1} \cup \{u_{2k} v_{2k}\}$. The resulting graph, $G - S_{3k+2}$ a single cycle remains, so isolating any vertex requires removing two additional edges. This section is summarized in $M_3(n,k,r) = 2r - k + 1 - \left\lfloor \frac{2(r-3k+1)}{3} \right\rfloor$ for $3k \leq r \leq 3k+3$.

Since $G - S_{3k+3}$ is a path of length $k+2$, only one edge needs to be removed to create an additional connected component. Thus, $M_3(n,k,r) = n + r$ for $3k+4 \leq r \leq 2n$. $\square$

Theorem 3.6. *Let n and k be two relatively prime integers, $n \geq 5$, such that $k = \min\{l : GP(n,l) \cong GP(n,k)\}$. If $n = 2k+3$, then for every integer $r = 2, \ldots, 2n$, $\lambda_r(GP(n,k)) \leq M_4(n,k,r)$, where*

$$M_4(n,k,r) = \begin{cases} 2r-1 & 2 \leq r \leq 7 \\ 2r - 2 - \left\lfloor \frac{r-8}{4} \right\rfloor & 8 \leq r \leq \min\{16, 3k-2\} \\ 2r - 4 - \left\lfloor \frac{r-16}{3} \right\rfloor & 17 \leq r \leq 3k-2 \\ 2r - k + 2 - \left\lfloor \frac{2}{3}(r - 3k + 2) \right\rfloor & 3k-1 \leq r \leq 3k+5 = 2n-k-1 \\ n + r & 2n - k \leq r \leq 2n \end{cases}$$

Proof. In the hypothesis of this theorem, the shortest cycles in $GP(n,k)$ have the form $C_i = \langle u_i, u_{i+1}, u_{i+2}, u_{i+3}, v_{i+3}, v_{i+k+3}, v_{i+2k+3 \equiv i}, u_i \rangle$, as seen in Figure 22.

We begin by isolating each vertex in C_0. It requires and edge cut of $S_2 = 3$ edges to

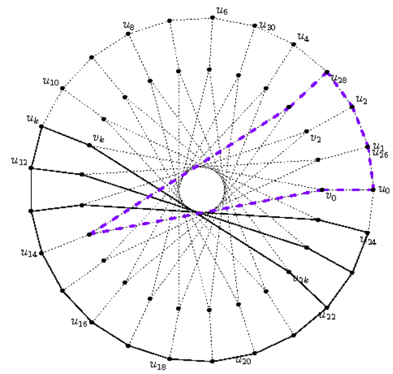


Figure 23. $GP(25, 11) - S_{3k-2}$

isolate u_0, and two additional edges to isolate each of u_1, u_2, u_3, v_3 and $v_k + 3$. Thus, $M_4(n, k, r) = 2r - 1$ for $2 \leq r \leq 7$. The final vertex of C_0 is v_0 and its isolation requires only one additional edge in the edge cut S_8, so $M_4(n, k, 8) = 2r - 2$.

The next copy of the cycle is C_1. The vertices u_1, u_2 and u_3 are isolated in $G - S_8$, so only four vertices are in the connected component. The first three of these vertices (u_4, v_4, v_{k+4}) require the removal of two additional edges to be isolated, so $M_4(n, k, r) = 2r - 2$ for $8 \leq r \leq 11$. The final vertex, v_1 is attached by a bridge so $M_4(n, k, 12) = 2r - 3$.

We proceed similarly for the cycle C_2. The vertices u_2, u_3 and u_4 are isolated in $G - S_{12}$, so four vertices are in the connected component. The first three of these vertices (u_5, v_5, v_{k+5}) require two additional edges, so $M_4(n, k, r) = 2r - 3$ for $12 \leq r \leq 15$. The final vertex, v_2 is attached by a bridge so $M_4(n, k, 12) = 2r - 4$. We summarize the last two paragraphs in $M_4(n, k, r) = 2r - 2 - \lfloor \frac{r-8}{4} \rfloor$ for $8 \leq r \leq \min 16, 3k - 2$.

Next, is C_3. In $G - S_{16}$, u_3, v_3, u_4 and u_5 are isolated vertices while v_{k+6}, v_6 and u_6 are not. Therefore, S_{17} gains two additional edges, S_{18} gains two additional edges, and S_{19} requires one additional edge. This pattern continues with C_4 through C_{k-4}. When all vertices in C_{k-4} have been isolated there are $3(k - 4 - 2) + 4 + 4 + 7 + 1 = 3k - 2$ components. Therefore, $M_4(n, k, r) = 2r - 4 - \lfloor \frac{r-16}{3} \rfloor$ for $17 \leq r \leq 3k - 2$.

$G - S_{3k-2}$ has the vertices $u_0, v_0, u_1, v_1, \ldots, u_{k-1}, v_{k-1}$ and $v_{k+3}, v_{k+4}, \ldots, v_{2k-1}$ already isolated, as seen in Figure 23. As we continue and isolate u_k, two additional edges must be removed, so $S_{3k-1} = S_{3k-2} \cup \{u_k u_{k+1}, u_k v_k\}$. In $G - S_{3k-1}$, the vertex v_k is connected by a bridge, so $S_{3k} = S_{3k-1} \cup \{v_k v_{2k}\}$. In $G - S_{3k}$, the vertex v_{2k} is also connected by a bridge, so $S_{3k+1} = S_{3k} \cup \{u_{2k} v_{2k}\}$. This pattern repeats with u_{k+1}, v_{k+1} and v_{2k+1}, so that the resulting graph, $G - S_{3k+4}$ a single cycle remains, so isolating any vertex requires removing two additional edges. This section is summarized in $M_4(n, k, r) = 2r - k + 2 - \left\lfloor \frac{2(r-3k+2)}{3} \right\rfloor$ for $3k - 1 \leq r \leq 3k + 5 = 2n - k - 1$.

$G - S_{3k+5}$ is a path of length $k + 2$. As such, only one edge needs to be removed for each additional component. Thus, $M_3(n, k, r) = n + r$ for $3k + 6 = 2n - k \leq r \leq 2n$. □

Theorem 3.7. *Let n and k be two relatively prime integers, $n \geq 5$, such that $k = \min\{l : GP(n, l) \cong GP(n, k)\}$. If $n = 3k + 2$, then for every integer $r = 2, \ldots, 2n$, $\lambda_r(GP(n, k)) \leq M_5(n, k, r)$, where*

$$M_5(n, k, r) = \begin{cases} 2r - 1 & 2 \leq r \leq 7 \\ 2r - \lfloor \frac{r-8}{4} \rfloor & 8 \leq r \leq \min\left\{16, \frac{9k-7}{2}\right\} \\ 2r - 4 - \lfloor \frac{r-16}{3} \rfloor & 17 \leq r \leq \frac{9k-7}{2} \\ 2r - \frac{3(k-3)}{2} - 2 - \lfloor \frac{2}{3}(r - \frac{9k-7}{2}) \rfloor & \frac{9k-5}{2} \leq r \leq \frac{9k+7}{2} \\ n + r & \frac{9k+9}{2} \leq r \leq 2n \end{cases}$$

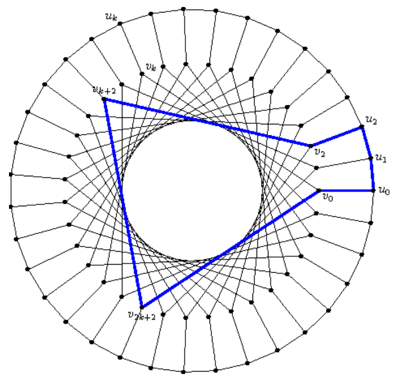


Figure 24. $GP(35, 11)$ which has girth 7, and $35 = 3 \cdot 11 + 2$. One copy of the smallest cycle is bold and in blue.

Proof. Graphs satisfying the conditions in the hypothesis have girth 7 and all shortest cycles are in the form $C_i = \langle u_i, u_{i+1}, u_{i+2}, v_{i+2}, v_{i+k+2}, v_{i+2k+2}, v_i, u_i\rangle$, as seen in Figure 24. Notice that $GP(n, k) \cong GP(n, l)$ if $l = \frac{n-3}{2} = \frac{3k-1}{2}$. Since $GP(n, l)$ satisfies all of the conditions for the previous theorem, except the minimality of the isomorphism class, we will use the previous case as a guideline, and will often use l in our calculations to simplify the expressions.

We begin by isolating the vertices of C_0. This is identical to the previous cases, so $M_5(n, k, r) = 2r - 1$ for $2 \le r \le 7$. Next, we will isolate the vertices of C_k. This is because $C_0 \cap C_k$ contains three vertices, whereas $C_0 \cap C_1$ only contains two. In addition, C_k corresponds to the "next" cycle in $GP(n, l)$. As we proceed through C_k we observe that in $G - S_7$ the vertices v_0, v_{k+2}, v_{2k+2} are already isolated. Thus, only four vertices v_k, u_k, u_{k+1} and u_{k+2} are in the connected component. It requires two additional edges for each edge cut, except the final vertex, so $|S_r| = M_5(n, k, r) = 2k - 2 - \left\lfloor \frac{r-8}{4} \right\rfloor$ for $8 \le r \le 12$. $G - S_{12}$ has isolated the vertices v_0, v_k and v_{2k+2} of C_{2k}. The vertices v_{2k}, u_{2k}, u_{2k+1} and u_{2k+2} remain in the connected component, so the pattern continues with $M_5(n, k, r) = 2k - 2 - \left\lfloor \frac{r-8}{4} \right\rfloor$ for $8 \le r \le 16$.

Next we proceed to C_{3k}. In $G - S_{16}$, four of the vertices of C_{3k} are isolated, u_0, v_0, v_k and v_{2k}. The function changes here so that two vertices v_{3k} and u_{3k} are isolated by adding two new edges to each respective edge cut, and u_{3k+1} gains only one additional edge. So, $M_5(n, k, r) = 2r - 4 - \left\lfloor \frac{r-16}{3} \right\rfloor$ for $17 \le r \le 19$.

We continue along the cycles whose subindices are multiples of k, to $C_{4k \equiv k-2 \bmod n}$. Since the vertices in C_k were previously isolated and C_k shares several vertices with C_{k-2}, there are only three vertices to isolate in $G - S_{19}$. This pattern continues until C_{mk} where $mk \equiv 3 \bmod n$. This condition is equivalent to $m = 3\frac{k-3}{2} + 1$. The pattern changes at C_3 because at this point u_3, u_4 and v_3, each has exactly two incident edges, so it only take 4 edges instead of 5 to isolate them. When the vertices of the cycles $C_0, C_k, C_{2k}, \ldots, C_{(m-1)k}$ are all isolated, there are $3(m) + 2 + 4 + 1 = 3\left(\frac{3(k-3)}{2} + 1\right) + 7 = \frac{9k-7}{2}$ connected components, where the $3(m)$ are 3 vertices for each cycle, the 2 comes from the extra vertices in C_k and C_{2k}, the 4 comes from the extra vertices in C_0, and the 1 come from the connected component. Thus, $M_5(n, k, r) = 2r - 4 - \left\lfloor \frac{r-16}{3} \right\rfloor$ for $17 \le r \le \frac{9k-9}{2}$. One valuable observation is that in $GP(n, l)$, the cutoff would be $3l - 2 = 3\left(\frac{3k-1}{2}\right) - 2 = \frac{9k-7}{2}$.

When the pattern changes at C_3, $S_{\frac{9k-5}{2}}$ gains two new edges, and $S_{\frac{9k-3}{2}}$ and $S_{\frac{9k-1}{2}}$ gains one new edge per vertex, see Figure 25. This pattern continues for C_{k+3}. Thus, $M_5(n, k, r) = 2r - \frac{3(k-3)}{2} - 2 - \left\lfloor \frac{2}{3}\left(r - \frac{9k-7}{2}\right) \right\rfloor$ for $\frac{9k-5}{2} \le r \le \frac{9k+7}{2}$.

At this point the only non-trivial connected component is a path. Thus, $M_5(n, k, r) = n + r$ for $\frac{9k+9}{2} \le r \le 2n$. □

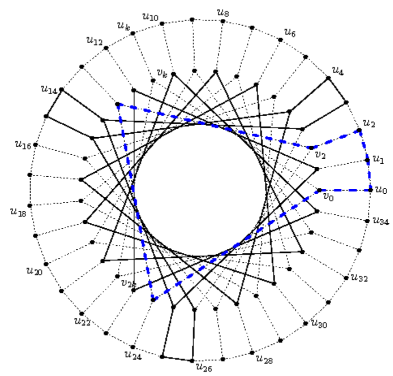


Figure 25. $GP(35, 11)$ which has girth 7, and $35 = 3 \cdot 11 + 2$.

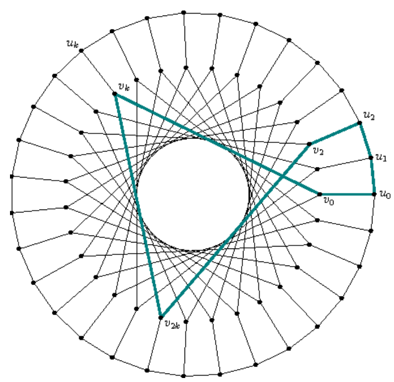


Figure 26. $GP(31, 11)$ which has girth 7, and $35 = 3 \cdot 11 + 2$. One copy of the shortest cycle is bold and in cyan.

Theorem 3.8. *Let n and k be two relatively prime integers, $n \geq 5$, such that $k = \min\{l : GP(n,l) \cong GP(n,k)\}$. If $n = 3k - 2$, then for every integer $r = 2, \ldots, 2n$, $\lambda_r(GP(n,k)) \leq M_6(n,k,r)$, where*

$$M_6(n,k,r) = \begin{cases} 2r - 1 & 2 \leq r \leq 7 \\ 2r - 2 - \left\lfloor \frac{r-8}{4} \right\rfloor & 8 \leq r \leq \min\left\{16, \frac{9k-19}{2}\right\} \\ 2r - 4 - \left\lfloor \frac{r-16}{3} \right\rfloor & 17 \leq r \leq \frac{9k-19}{2} \\ 2r - \frac{3k-9}{2} - \left\lfloor \frac{2}{3}(r - \frac{9k-17}{2}) \right\rfloor & \frac{9k-17}{2} \leq r \leq \frac{9k-5}{2} \\ n + r & \frac{9k-3}{2} \leq r \leq 2n \end{cases}$$

Proof. Graphs satisfying the hypothesis of this theorem have girth 7, so the shortest cycles are in the form $C_i = \langle u_i, u_{i+1}, u_{i+2}, v_{i+2}, v_{i+2k}, v_{i+k}, v_i, u_i \rangle$, as seen in Figure 26. If we were to invert $GP(n,k)$ into its isomorphic partner $GP(n,l)$ where $l = \frac{n-3}{2} = \frac{3k-5}{2}$, then $GP(n,l)$ satisfies all of the conditions for the previous case, except the minimality of the isomorphism class.

We begin by isolating the vertices of C_0. This is identical to the previous cases, so $M_6(n,k,r) = 2r - 1$ for $2 \leq r \leq 7$. Next, we will isolate the vertices of C_k. Again, this is because $C_0 \cap C_k$ contains three vertices, whereas $C_0 \cap C_1$ only contains two. As we proceed through C_k we observe that in $G - S_7$ the vertices v_2, v_k, v_{2k} are already isolated. Thus, only four vertices v_{k+2}, u_k, u_{k+1} and u_{k+2} are in the connected component. It requires two additional edges for each edge cut, except the final vertex, so $|S_r| = M_6(n,k,r) = 2k - 2 - \left\lfloor \frac{r-8}{4} \right\rfloor$ for $8 \leq r \leq 12$. $G - S_{12}$ has isolated the vertices v_2, v_{k+2} and v_{2k} of C_{2k}. The vertices $v_{2k+2}, u_{2k}, u_{2k+1}$ and u_{2k+2} remain in the connected component, so the

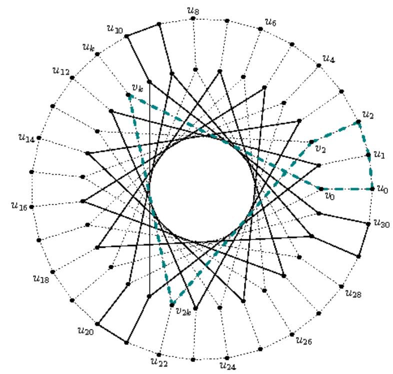


Figure 27. $GP(31,11) - S_{\frac{9k-19}{2}=40}$. This shows what happens at $C_{n-3=28}$

pattern continues with $M_6(n,k,r) = 2k-2-\lfloor\frac{r-8}{4}\rfloor$ for $8 \leq r \leq 16$.

Next we proceed to $C_{3k\equiv 2 \bmod n}$. In $G - S_{16}$, four of the vertices of C_2 are isolated, u_2, v_2, v_{k+2} and v_{2k+2}. The function changes here so that two vertices u_3 and u_4 are isolated by adding two new edges to each respective edge cut, and v_4 gains only one additional edge. So, $M_6(n,k,r) = 2r-4-\lfloor\frac{r-16}{3}\rfloor$ for $17 \leq r \leq 19$.

We continue along the cycles whose subindices are multiples of k, to $C_{4k\equiv k-2 \bmod n}$. Since all vertices in C_k were previously isolated and C_k shares several vertices with C_{k+2}, there are only three vertices to isolate in $G-S_{19}$. This pattern continues until C_{mk} where $mk \equiv -3 \bmod n$. This condition is equivalent to $m = 3\frac{k-3}{2} - 1$. The pattern changes at $C_{mk\equiv n-3 \bmod n}$ because at this point u_{n-2}, u_{n-1} and v_{n-1} each has exactly two incident edges, so it only take 4 edges instead of 5 to isolate them. When the vertices of the cycles $C_0, C_k, C_{2k}, \ldots, C_{(m-1)k}$ are all isolated, there are $3(m)+2+4+1 = 3\left(\frac{3(k-3)}{2}-1\right)+7 = \frac{9k-21}{2}$ components, where the $3(m)$ are 3 vertices for each cycle, the 2 comes from the extra vertices in C_k and C_{2k}, the 4 comes from the extra vertices in C_0, and the 1 come from the connected component. Thus, $M_6(n,k,r) = 2r-4-\lfloor\frac{r-16}{3}\rfloor$ for $17 \leq r \leq \frac{9k-19}{2}$.

When the pattern changes at C_{n-3}, it changes so that $S_{\frac{9k-17}{2}}$ gains two new edges, and $S_{\frac{9k-15}{2}}$ and $S_{\frac{9k-13}{2}}$ gains one new edge per vertex(see Figure 27. This pattern continues for C_{k-3}. Thus, $M_6(n,k,r) = 2r - \frac{3k-9}{2} - \lfloor\frac{2}{3}(r-\frac{9k-17}{2})\rfloor$ for $\frac{9k-17}{2} \leq r \leq \frac{9k-5}{2}$.

At this point, the only non-trivial connected component is a path. Thus, $M_6(n,k,r) = n+r$ for $\frac{9k-3}{2} \leq r \leq 2n$. □

4. Conclusion and Further Research

In this paper we present upper and lower bounds for the r component edge connectivity of generalized Petersen graphs. However, we left open the identification of the graphs where one of those bounds provide the exact value of the r-component edge connectivity.

In the introduction we defined the edge and the vertex versions of the concept of component connectivity. In this paper we have only studied the r-component edge connectivity. It is also interesting to study the vertex r-component connectivity of generalized Petersen graphs.

Generalized Petersen graphs are an interesting family of vertex-transitive and 3-regular graphs. It would be interesting to explore the extension of the results we found to other vertex-transitive graphs, or to other families of regular graphs.

Finally, it would be interesting to study the notion of r-component connectivity with the additional condition that the order of the connected components must be the same, or as close to each other as possible. This is a much hard problem, and a more relaxed and also interesting version of the problem would be to add the condition that the connected components cannot be trivial. That is, to combine the concepts of super connectivity and component connectivity.

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