Connectivity in Directed Hypergraphs

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Abstract. This paper concentrates on the relation between the connectivity, the diameter, and some new parameters that we introduce. Such parameters are closely related with other ones introduced by Fàbrega and Fiol [2] to study the connectivity of digraphs. Some interesting results on Kautz and De Bruijn bus networks are obtained.

1 Introduction

Bus networks are usually modeled by hypergraphs, directed or not, depending on the nature bidirectional or unidirectional of the buses. In this paper we concentrate in directed hypergraphs, also called hyperdigraphs, for short. A *hyperdigraph* H is a pair $(\mathcal{V}(H), \mathcal{E}(H))$, where $\mathcal{V}(H)$ is a non-empty set of *vertices* or *nodes*, and $\mathcal{E}(H)$ is a set of ordered pairs of non-empty subsets of $\mathcal{V}(H)$, called *buses*. If $E = (E^-, E^+)$ is a bus, we say that E^- is the *in-set*, E^+ is the *out-set* of E, and that E joins vertices in E^- to vertices in E^+. Its *in-size(out-size)* is the cardinal of E^-, $|E^-|(|E^+|)$. If v is a vertex, the *in-degree(out-degree)* of v is the number of buses containing v in the out-set(in-set), and it is denoted by $d^-(v)(d^+(v))$.

If H is a directed bus network, its *order* is the number of vertices, $|\mathcal{V}(H)|$, denoted by $n(H)$, and its *size*, is the number of buses, $m(H)$. The *maximum in-size* and *maximum out-size* of H are respectively defined by

$$s^-(H) = \max\{|E^-| : E \in \mathcal{E}(H)\}\ ,\ s^+(H) = \max\{|E^+| : E \in \mathcal{E}(H)\}$$

Similarly, the *maximum in-degree* , *maximum out-degree* of H are

$$d^-(H) = \max\{d^-(v) : v \in \mathcal{V}(H)\}\ ,\ d^+(H) = \max\{d^+(v) : v \in \mathcal{V}(H)\}$$

We denote $s(H) = \max\{s^+(H), s^-(H)\}$, $d(H) = \max\{d^+(H), d^-(H)\}$. Note that when $s = 1$, H is a digraph.

A *path* of *length* k from a vertex u to a vertex v in H is an alternating sequence of vertices and buses $u = v_0, E_1, v_1, E_2, v_2, \ldots, E_k, v_k = v$ such that $v_i \in E^-_{i+1}$, $(i = 0, \ldots, k-1)$ and $v_i \in E^+_i$, $(i = 1, \ldots, k)$. The *distance* from u to v, $d(u, v)$, is the length of the shortest path from u to v. The *diameter*, $D(H)$, is the maximum distance between every pair of vertices of H. Any two paths in H are *vertex-disjoint* if they have no internal vertices in common, and are *bus-disjoint* if they do not share buses.

A hyperdigraph is *connected* if there exists at least one path from each vertex to any other vertex. The *vertex-connectivity*, $\kappa(H)$, of a hyperdigraph H, is the minimum number of vertices to be removed to obtain a non-connected or trivial hyperdigraph (a hyperdigraph with only one vertex). Similarly is defined the *bus-connectivity*, $\lambda(H)$.

The *dual hyperdigraph*, H^*, of a hyperdigraph H has its set of vertices in one-to-one correspondence with the set of buses of H, and for every vertex v of H it has a bus, (V^-, V^+), such that a vertex $e \in V^-$ if and only if $v \in E^+$ and $e \in V^+$ if and only if $v \in E^-$.

The *underlying digraph* of a hyperdigraph H is the digraph $\widehat{H} = (\mathcal{V}(\widehat{H}), \mathcal{A}(\widehat{H}))$ with $\mathcal{V}(\widehat{H}) = \mathcal{V}(H)$ and $\mathcal{A}(\widehat{H}) = \{(u,v) : \exists E \in \mathcal{E}(H), u \in E^-, v \in E^+\}$. That is, there is an arc from a vertex u to a vertex v in $\widehat{H}$ if and only if there is a bus joining u to v in H. So, paths in $\widehat{H}$ and H are in correspondence, and this implies $D(\widehat{H}) = D(H)$ and $\kappa(\widehat{H}) = \kappa(H)$. We are going to denote $\widehat{\kappa} = \kappa(\widehat{H})$ and $\widehat{\lambda} = \lambda(\widehat{H})$.

In [2] is introduced a parameter l_π for digraphs. Bounds for the vertex and arc-connectivities are obtained in terms of it.

In this paper two variations of parameter l_π, for hyperdigraphs, are introduced in Section 2. In Section 3 there are bounds on the connectivity. These are applied to the De Bruijn and Kautz bus networks in Section 4.

2 Parameters ℓ_π^v and ℓ_π^h

Definition 2.1. *Let H be a hyperdigraph with minimum degree d, minimum size s and diameter D. For an integer π, $0 \le \pi \le ds - 2$, let $\ell_\pi^v = \ell_\pi^v(H)$, $1 \le \ell_\pi^v \le D$, be the greatest integer such that, for any two (not necessarily different) vertices $x, y \in \mathcal{V}(H)$,*

- *a) if $d(x,y) < \ell_\pi^v$, there are not two shortest paths vertex-disjoint from x to y and there are at most π vertex-disjoint paths from x to y of length $d(x,y)+1$;*
- *b) if $d(x,y) = \ell_\pi^v$, there are not two shortest paths vertex-disjoint from x to y.*

Definition 2.2. *Let H be a hyperdigraph with minimum degree d, minimum size s and diameter D. For an integer π, $0 \le \pi \le ds - 2$, let $\ell_\pi^h = \ell_\pi^h(H)$, $1 \le \ell_\pi^h \le D$, be the greatest integer such that, for any two (not necessarily different) vertices $x, y \in \mathcal{V}(H)$,*

a) *if $d(x,y) < \ell^h_\pi$, there are no two shortest paths bus-disjoint from x to y and there are at most π bus-disjoint paths x to y of length $d(x,y)+1$;*

b) *if $d(x,y) = \ell^h_\pi$, there are no two shortest bus-disjoint paths from x to y.*

Proposition 2.1. *Let $H = (\mathcal{V}(H), \mathcal{E}(H))$ be a hyperdigraph. Then,*

a) $\ell^v_\pi(H) = \ell_\pi(\widehat{H})$;

b) $\ell^h_\pi(H) = \ell_\pi(\widehat{H^*})$. □

3 Bounds on connectivity

Theorem 3.1. *Let $H = (\mathcal{V}(H), \mathcal{E}(H))$ be a hyperdigraph with minimum degree d, minimum size s, $D = D(H)$ and vertex-connectivity κ. For any π, $0 \le \pi \le ds - 2$, let $\ell^v_\pi = \ell^v_\pi(H)$. Then, $\kappa \ge ds - \pi$, if $D \le 2\ell^v_\pi - 1$.*

Proof: Let $F \in \mathcal{V}(H)$ be a minimum disconnecting set of H, i.e., $|F| = \kappa$ and $H - F$ is not connected. Then the set $V - F$ is partitioned into two disjoint nonempty sets V^- and V^+ such that $H - F$ has buses from V^- to V^+. The vertices of V^- and V^+ are respectively partitioned into subsets $V_i = \{x : d(x,F) = i\}$, $1 \le i \le k$ and $V'_j = \{x : d(F,x) = j\}$, $1 \le j \le k'$. As any path from V^- to V^+ goes through F, the distance from a vertex in V_k to one in V'_k is at least $k + k'$, so $k + k' \le D$. Since $D \le 2\ell^v_\pi - 1$ one of them, k or k' is at most $\ell^v_\pi - 1$. Without lost of generality, suppose $k \le k'$. Hence, $k \le \ell^v_\pi - 1$.

Now let x be a vertex in V_k such that the set F' of vertices of F at distance exactly k from x has minimum cardinal. For each $f \in F$ such that $d(x,f) \le k+1$, we consider the shortest path from x to f. Let Y be the set of vertices adjacent from x such that neither of these paths cross it: $ds \le ds^+(x) \le |Y| + |F|$. Let $y \in Y$, then $d(y,f) > k+1$, for all $f \in F - F'$. Since $|F'|$ is minimum, for all $y \in Y$, $f \in F'$, there exists a path from y to f of length k. Besides $d(y,f) \ge k$, because if not, there would be two vertex-disjoint paths from x to f, for some $f \in F'$ of length k or $k+1$, contradicting the definition of ℓ^v_π. Therefore, $|Y| \le \pi$ and $|F| \ge ds - \pi$. □

Theorem 3.2. *Let $H = (\mathcal{V}(H), \mathcal{E}(H))$ be a hyperdigraph with minimum degree d, minimum size s, $D = D(H)$ and bus-connectivity λ. For any π, $0 \le \pi \le ds - 2$, let $\ell^h_\pi = \ell^h_\pi(H)$. Then, $\lambda \ge ds - \pi$, if $D \le 2\ell^h_\pi$.* □

This theorem can be proved in the same way that Theorem 3.1.

4 Applications

In [1] were introduced the De Bruijn and Kautz bus networks There it was shown that they have good order in relation to their maximum vertex degree and bus size We recall here the following property of [1]

Let $GB(d,n,s,m)$ be a De Bruijn bus network with degree d, order n, size s and m hyperarcs, and let $GK(d,n,s,m)$ be the Kautz bus network with degree d, order n, size s and m hyperarcs Then,

a) $\widehat{GB}(d,n,s,m) = RPK(ds,n)$, the generalized De Bruijn digraph [4],

b) $\widehat{GK}(d,n,s,m) = II(ds,n)$, the generalized Kautz digraph in [3] □

Theorem 4 1. *Let $GB(d,n,s,m)$ and $GK(d,n,s,m)$ be respectively the De Bruijn and Kautz bus networks with degree d, order n, bus size s and m buses Then,*

a) $H = GK(d,n,s,m) \Longleftrightarrow H^* = GK(s,m,d,n)$,

b) $H = GB(d,n,s,m) \Longleftrightarrow H^* = GB(s,m,d,n)$

Proof a) Since $H = H^{**}$ is enough to prove that if $H = GK(d,n,s,m)$ then $H^* = GK(s,m,d,n)$ Applying the definition of the dual hyperdigraph to H, clearly appears H^* Similarly can be proved b) □

Corollary 4.2. *Let $GB(d,n,s,m)$ and $GK(d,n,s,m)$ be respectively the De Bruijn nad Kautz bus networks with degree d, order n, bus size s and m buses Let $H = GK(d,n,s,m)$ or $H = GB(d,n,s,m)$ Then,*

$$\kappa(H) \geq ds - 1, \text{ if } D(H) \geq 3 \text{ and } \lambda(H) \geq d - 1, \text{ if } D(H) \geq 2$$

Proof By Proposition 2 1, $\ell_1^v(H) = \ell_1(\widehat{H}) \geq D(H) - 1$, $\ell_1^b(H) = \ell_1(\widehat{H^*}) \geq D(H) - 1$ [2] Then, we apply Theorems 3 1 and 3 2 with these values □

References

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