Diameter of path graphs

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Abstract

For a given graph $G$ and a positive integer $k$, the $k$-path graph, $P_k(G)$, has for vertices
the set of all paths of length $k$ in $G$. Two vertices are adjacent when the intersection of the
respective paths forms a path of length $k-1$ in $G$, and their union forms either a cycle
or a path of length $k+1$ in $G$. Path graphs were previously studied and more generally, some bounds have been presented in
the case $k \leq 5$. In this paper we present upper bounds for the diameter of iterated $k$-path
graphs for any positive integer $k$, which improve the known upper bounds.

Keywords: Path graph, diameter, girth, degree.

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1. Introduction

The $k$-path graph corresponding to a graph $G$ has for vertices the set of all paths of length $k$ in $G$. Two vertices are connected by an edge whenever the intersection of the corresponding paths forms a path of length $k - 1$ in $G$, and their union forms either a cycle or a path of length $k + 1$ in $G$. Intuitively, this means that the vertices are adjacent if and only if one can be obtained from the other by “shifting” the corresponding paths in $G$. Following the notation used by Knor and Niepel, the $k$-path graph of $G$ will be denoted as $P_k(G)$. Path graphs were introduced by Broersma and Hoede in [3] as a natural generalization of line graphs. A characterization of $P_2$-path graphs is given in [3] and [8]; some important structural properties of path graphs are presented in [10], [11], [12] and [1]. Distance properties of path graphs are studied in [2] and in [6]. The edge connectivity and super edge-connectivity of line graphs was studied by Jixiang Meng in [13]. The connectivity of path graphs was studied by Xueliang Li [9], later by Knor, Niepel [5, 7], and Mallah [7] and more recently by Balbuena [2], and Ferrero [2, 5, 6].

2. Definitions, notation and previous results

Let $G = (V, E)$ be a simple graph, i.e. with no loops or multiple edges, with vertex set $V(G)$ and edges $E(G)$. The neighbourhood of a vertex $v$ is the set $N(v)$, of all vertices adjacent with $v$. The degree of a vertex $v$ is $\deg(v) = |N(v)|$. The minimum degree of the graph $\delta(G)$, is the minimum degree over all vertices of $G$.

A path of length $n$ in $G$ between two vertices $u$ and $v$ is a sequence of vertices $u = x_0, x_1, \ldots, x_n = v$ where $(x_i, x_{i+1})$ is an edge for $i = 0, \ldots, n - 1$. A graph $G$ is called connected if every pair of vertices is joined by a path. A cycle in $G$ is a path $u = x_0, x_1, \ldots, x_n = u$. The girth of a graph $G$ is denoted by $g(G)$ and it is the length of a shortest cycle in $G$.

The distance between two vertices in $G$ is the length of a shortest path joining them. Then, the diameter of a graph $G$, denoted as $D(G)$, is the maximum distance between any two vertices of $G$. Notice that for a disconnected graph $G$, $D(G) = \infty$.

The $k$-path graph corresponding to a graph $G$ is denoted as $P_k(G)$ and has for vertices the set of all paths of length $k$ in $G$. Two vertices
are connected by an edge whenever the intersection of the corresponding
paths forms a path of length $k - 1$ in $G$, and their union forms either a
cycle or a path of length $k + 1$ in $G$. The connectivity of $k$-path graphs
was previously studied by Knor and Niepel [5]. They introduced some
notation to formulate an important result that we recall next.

For a graph $G$ and two integers $k$ and $t$, $k \geq 2$ and $0 \leq t \leq k - 2$,
$p^*_{k,t}$ denotes an induced tree in $G$ with diameter $k + t$ and a diametric
path $(x_0, x_1, \ldots, x_t, v_0, v_1, \ldots, v_{k-t-1}, y_1, y_2, \ldots, y_t)$ such that all the end
vertices of $p^*_{k,t}$ are at distance no greater than $t$ from $v_0$ or $v_{k-t}$,
the degrees of $v_1, v_2, \ldots, v_{k-t-1}$ are $2$ in $p^*_{k,t}$ and no vertex in $V(p^*_{k,t}) -
\{v_1, v_2, \ldots, v_{k-t-1}\}$ is adjacent with a vertex in $V(G) - V(p^*_{k,t})$. The path
$v_1, v_2, \ldots, v_{k-t-1}$ is the base of $p^*_{k,t}$, and for a path $A$ of length $k$ we say
that $A \in p^*_{k,t}$ if and only if the base of $p^*_{k,t}$ is a subpath of $A$.

**Theorem A (5).** Let $G$ be a connected graph with girth at least $k + 1$. Then,
p$_k(G)$ is disconnected if and only if $G$ contains a $p^*_{k,t}$, $0 \leq t \leq k - 2$, and a
path $A$ of length $k$, such that $A \notin p^*_{k,t}$.

Given a simple graph $G$ and a path of length $k$ in it, let us say
$u_0, u_1, \ldots, u_k$, we are going to denote the vertex in $p_k(G)$ by $U =
u_0 u_1 \ldots u_k$.

3. Diameter of $k$-path graphs

The study of the diameter of a graph is interesting for connected
graphs, Knor and Niepel [5] provided in Theorem A a characterization of
connected $k$-path graphs $p_k(G)$ of graphs with large girth. In this section,
we present upper bounds for the diameter of those graphs.

**Lemma 3.1.** Let $k$ be a positive integer and let $G$ be a graph with minimum
degree $\delta \geq 2$, girth $g \geq k + 1$ and such that $p_k(G)$ is connected. If $U$ and
$V$ are two vertices in $p_k(G)$ determined by paths in $G$ with share an endvertex,
then there is a path of length $2k$ joining $U$ and $V$.

**Proof.** Let the vertices $U$ and $V$ be determined by the paths $U =
u_0 u_1 \ldots u_k$ and $V = v_0 v_1 \ldots v_k$ in $G$. Since the paths $u_0, u_1, \ldots, u_k$
and $v_0, v_1, \ldots, v_k$ share an endvertex, without loss of generality we can
assume $u_0 = v_0$. Since $\delta \geq 2$ and $g \geq k + 1$, there exists a vertex
$x_k \in N(u_0) \setminus \{u_1, \ldots, u_{k-1}, v_1, \ldots, v_{k-1}\}$, and as a consequence, there
exist vertices $U_1 = x_k u_0 u_1 \ldots u_{k-1}$ and $V_1 = x_k v_0 v_1 \ldots v_{k-1}$ in $p_k(G)$,
$U_1 \in N(U)$ and $V_1 \in N(V)$. Notice that the paths $u_0, u_1, \ldots, u_k$ and
\(v_0, v_1, \ldots, v_k\) could eventually share more vertices than an endvertex. Indeed, we can repeat the previous procedure with \(U_1\) and \(V_1\) and obtain a vertex \(x_{k-1} \in N(x_k) \setminus \{u_0, \ldots, u_{k-2}, v_0, \ldots, v_{k-2}\}\) and vertices \(U_2 = x_{k-1}x_{k-2}u_1 \ldots u_{k-2}\) and \(V_2 = x_{k-1}x_{k-2}v_1 \ldots v_{k-2}\) in \(P_G(G)\), \(U_2 \in N(U_1)\) and \(V_2 \in N(V_1)\). Repeating this procedure \(k\) times we will obtain two paths in \(P_G(G)\), \(U_k \ldots U_{k-1}\) and \(V_k \ldots V_{k-1}\), where \(U_k = x_1 \ldots x_{k-1}\) and \(V_k = x_1 \ldots x_{k-1}\). Then, \(U_k = V_k\) because \(u_0 = v_0\) and we have a path in \(P_G(G)\), the path \(U, U_1, \ldots, U_{k-1}, U_k, V_{k-1}, \ldots, V_1, V\), joining \(U\) and \(V\). Clearly, the length of that path is \(2k\).

The above lemma can be extended to two vertices in \(P_G(G)\) whose corresponding paths in \(G\) share a vertex, which is not necessarily an endvertex. We are going to prove it, but to simplify the writing we first introduce some notation.

Let \(P\) be the path \(a_0, a_1, \ldots, a_r\) in \(G\). If \(r \geq k\) and no two vertices at distance smaller than or equal to \(k\) to \(P\) coincide, there exist vertices \(a_0a_1 \ldots a_k\) and \(a_{r-k+1}a_{r-k+2} \ldots a_r\) in \(P_G(G)\). Moreover, the path \(P\) induces a path in \(P_G(G)\) between them, which is going to be denoted as \(I_k(P)\) or equivalently, \(I_k(a_0, a_1, \ldots, a_r)\). Note that \(I_k(P)\) has length \(r - k\).

Lemma 3.2. Let \(k\) be a positive integer and let \(G\) be a graph with minimum degree \(\delta \geq 2\), girth \(g \geq k + 1\) and such that \(P_G(G)\) is connected. If \(U\) and \(V\) are two vertices in \(P_G(G)\) determined by paths in \(G\) which share a vertex, then there is a path joining \(U\) and \(V\). Moreover, such path has length at most \(2k\).

Proof. Let the vertices \(U\) and \(V\) be determined by the paths \(U = v_0u_1 \ldots u_k\) and \(V = v_0v_1 \ldots v_k\) in \(G\). If the paths \(u_0, u_1, \ldots, u_k\) and \(v_0, v_1, \ldots, v_k\) share an endvertex it suffices to apply Lemma 3.1. If not, there exist vertices \(u_0\) and \(v_0\) such that \(u_0 = v_0\) and \(\{u_0, \ldots, u_{k-1}\} \cap \{v_0, \ldots, v_{k-1}\} = \emptyset\). Without loss of generality we can assume \(s \geq t\). Then, proceeding as in the proof of Lemma 3.1, since \(\delta \geq 2\) and \(g \geq k + 1\) it is possible to construct a path \(x_{k-r}, \ldots, x_{k-1}, u_0\) which gives rise to the paths \(I_k(x_{k-r}, \ldots, x_{k-1}, u_0, \ldots, u_k)\) and \(I_k(x_{k-r}, \ldots, x_{k-1}, v_0, \ldots, v_{k-1}, v_k)\) in \(P_G(G)\). The union of those two paths determines a path in \(P_G(G)\) joining \(U\) and the vertex \(u_0, \ldots, u_{k-1}v_1 \ldots v_k\). At the same time, the vertex \(u_{k-t}, \ldots, u_{k-1}v_1 \ldots v_k\) is connected to \(V\). Indeed, if we proceed as in the proof of Lemma 3.1, since \(\delta \geq 2\) and \(g \geq k + 1\) we can find a path \(v_{k-r}, \ldots, v_{k-1}\) in \(G\) from which arise the paths \(I_k(u_{k-t}, \ldots, u_{k-2}v_1 \ldots v_k, v_0, \ldots, v_{k-1})\) and
Lemma 3.3. Let \( k \) be a positive integer and let \( G \) be a connected graph with minimum degree \( \delta \geq 2 \), girth \( g \geq k+1 \) and such that \( P_k(G) \) is connected. If \( U \) and \( V \) are two vertices in \( P_k(G) \) whose corresponding paths in \( G \) do not share any vertex, then there is a path of length at most \( 2k+D(G) \) joining \( U \) and \( V \).

Proof. Let the vertices \( U \) and \( V \) be determined by the paths \( U = u_0v_1...u_k \) and \( V = v_0v_1...v_k \) in \( G \). Let us assume that the shortest path between \( \{u_0,...,u_k\} \) and \( \{v_0,...,v_k\} \) is the shortest path between the vertices \( u_0 \) and \( v_0 \), \( u_k = z_0, z_1, ..., z_d = v_k \). Note that because of this choice, \( \{u_0,...,u_k\} \cap \{z_1,...,z_{d-1}\} = \emptyset \) and \( \{v_0,...,v_k\} \cap \{z_1,...,z_{d-1}\} = \emptyset \). Since \( \delta \geq 2 \) and \( g \geq k+1 \) there exist paths \( u_0,...,x_{d+1},u_0 \) and \( v_0,y_0,...,y_{d-1} \), in such a way that there are paths \( I_k(u_0,...,x_{d+1},u_0,...,u_k) \), \( I_k(x_0,...,x_{d+1},v_0,...,v_k) \), \( I_k(v_0,...,y_0,...,y_{d-1}) \). The union of those three paths forms a path joining \( U \) and \( V \). Besides, the lengths of those three paths are respectively \( k-s \), \( d+k-t \) and \( t \). Therefore, the total length will be \( 2k+d-s \). Since \( s \geq 0 \) and \( d \leq D(G) \), we conclude that the length of the path between \( U \) and \( V \) is at most \( 2k+D(G) \).

As a direct consequence of the previous lemmas we can obtain the following theorem regarding the diameter.

Theorem 3.4. Let \( k \) be a positive integer and let \( G \) be a graph with minimum degree \( \delta \geq 2 \), girth \( g \geq k+1 \) and such that \( P_k(G) \) is connected. Then, \( D(P_k(G)) \leq D(G) + 2k \).

The previous theorem complement the results from Knor and Niepel [6] and Belan and Jurica [2]. Moreover, they improve the upper bounds presented by Belan and Jurica for \( 2 \leq k \leq 4 \).

References


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