Diameter of path graphs*

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Abstract

For a given graph G and a positive integer k the k-path graph, $P_k(G)$, has for vertices the set of all paths of length k in G. Two vertices are adjacent when the intersection of the corresponding paths forms a path of length $k-1$ in G, and their union forms either a cycle or a path of length $k+1$ in G. Path graphs were proposed as an extension of line graphs. Indeed, $P_1(G)$ coincides with the line graph of G. The diameter of line graphs and 2-path graphs have been previously studied and more generally, some bounds have presented in the case $k \leq 5$. In this paper we present upper bounds for the diameter of iterated k-path graphs for any positive integer k, which improve the known upper bounds.

Keywords : Path graph, diameter, girth, degree.

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1. Introduction

The k-path graph corresponding to a graph G has for vertices the set of all paths of length k in G. Two vertices are connected by an edge whenever the intersection of the corresponding paths forms a path of length $k-1$ in G, and their union forms either a cycle or a path of length $k+1$ in G. Intuitively, this means that the vertices are adjacent if and only if one can be obtained from the other by "shifting" the corresponding paths in G. Following the notation used by Knor and Niepel, the k-path graph of G will be denoted as $P_k(G)$. Path graphs were introduced by Broersma and Hoede in [3] as a natural generalization of line graphs. A characterization of P_2-path graphs is given in [3] and [8], some important structural properties of path graphs are presented in [10], [11], [12] and [1]. Distance properties of path graphs are studied in [2] and in [6]. The edge connectivity and super edge-connectivity of line graphs was studied by Jixiang Meng in [13]. The connectivity of path graphs was studied by Xueliang Li [9], later by Knor, Niepel [5,7], and Mallah [7] and more recently by Balbuena [2], and Ferrero [2,5,6].

2. Definitions, notation and previous results

Let $G=(V,E)$ be a simple graph, i.e. with no loops or multiple edges, with vertex set $V(G)$ and edges $E(G)$. The *neighbourhood* of a vertex v is the set $N(v)$, of all vertices adjacent with v. The *degree* of a vertex v is $\deg(v)=|N(v)|$. The *minimum degree* of the graph $G\delta(G)$, is the minimum degree over all vertices of G.

A *path of length* n in G between two vertices u and v is a sequence of vertices $u-x_0,x_1,\ldots,x_n=v$ where (x_i,x_{i+1}) is an edge for $i=0,\ldots,n-1$. A graph G is called *connected* if every pair of vertices is joined by a path. A *cycle* in G is a path $u=x_0,x_1,\ldots,x_n=u$. The *girth* of a graph G is denoted by $g(G)$ and it is the length of a shortest cycle in G.

The *distance* between two vertices in G is the length of a shortest path joining them. Then, the *diameter* of a graph G, denoted as $D(G)$, is the maximum distance between any two vertices of G. Notice that for a disconnected graph G, $D(G)=\infty$.

The k-path graph corresponding to a graph G is denoted as $P_k(G)$ and has for vertices the set of all paths of length k in G. Two vertices are connected by an edge whenever the intersection of the corresponding paths forms a path of length $k-1$ in G, and their union forms either a cycle or a path of length $k+1$ in G. The connectivity of k-path graphs was previously studied by Knor and Niepel [5]. They introduced some notation to formulate an important result that we recall next.

For a graph G and two integers k and $t,k \geq 2$ and $0 \leq t \leq k-2$, $p^*_{k,t}$ denotes an induced tree in G with diameter $k+t$ and a diametric path $(x_t, x_{t-1}, \dots, x_1, v_0, v_1, \dots, v_{k-t}, y_1, y_2, \dots, y_t)$ such that all the end vertices of $P^*_{k,t}$ area at distance no greater that t from v_0 or v_{k-t}, the degrees of $v_1, v_2 \dots v_{k-t-1}$ are 2 in $P^*_{k,t}$ and no vertex in $V(P^*_{k,t}) - \{v_1, v_2 \dots v_{k-t-1}\}$ is adjacent with a vertex in $V(G) - V(P^*_{k,t})$. The path $v_1, v_2 \dots v_{k-t-1}$ is the *base* of $P^*_{k,t}$, and for a path A of length k we say that $A \in P^*_{k,t}$ if and only if the base of $P^*_{k,t}$ is a subpath of A.

Theorem A ([5]). *Let G be a connected graph with girth at least $k+1$. Then, $P_k(G)$ is disconnected if and only if G contains a $P^*_{k,t}$, $0 \leq t \leq k-2$, and a path A of length k, such that $A \notin P^*_{k,t}$.*

Given a simple graph G and a path of length k in it, let us say $u_0, u_1, \dots, u_k$, we are going to denote the vertex in $P_k(G)$ by $U = u_0u_1 \dots u_k$.

3. Diameter of k-path graphs

The study of the diameter of a graph is interesting for connected graphs, Knor and Niepel [5] provided in Theorem A a characterization of connected k-path graphs $P_k(G)$ of graphs with large girth. In this section, we present upper bounds for the diameter of those graphs.

Lemma 3.1. *Let k be a positive integer and let G be a graph with minimum degree $\delta \geq 2$, girth $g \geq k+1$ and such that $P_k(G)$ is connected. If U and V are two vertices in $P_k(G)$ determined by paths in G with share an endvertex, then there is a path of length $2k$ joining U and V.*

Proof. Let the vertices U and V be determined by the paths $U = u_0u_1 \dots u_k$ and $V = v_0v_1 \dots v_k$ in G. Since the paths $u_0, u_1, \dots, u_k$ and $v_0, v_1, \dots, v_k$ share an endvertex, without loss of generality we can assume $u_0 = v_0$. Since $\delta \geq 2$ and $g \geq k+1$, there exists a vertex $x_k \in N(u_0) \setminus \{u_1, \dots, u_{k-1}, v_1, \dots, v_{k-1}\}$, and as a consequence, there exist vertices $U_1 = x_ku_0u_1 \dots u_{k-1}$ and $V_1 = x_kv_0v_1 \dots v_{k-1}$ in $P_k(G)$, $U_1 \in N(U)$ and $V_1 \in N(V)$. Notice that the paths $u_0, u_1, \dots, u_k$ and

$v_0, v_1, \ldots, v_k$ could eventually share more vertices than an endvertex. Indeed, we can repeat the previous procedure with U_1 and V_1 and obtain a vertex $x_{k-1} \in N(x_k) \setminus \{u_0, \ldots, v_{k-2}, v_0, \ldots, v_{k-2}\}$ and vertices $U_2 = x_{k-1}x_ku_0u_1 \ldots u_{k-2}$ and $V_2 = x_{k-1}x_kv_0v_1 \ldots v_{k-2}$ in $P_k(G)$, $U_2 \in N(U_1)$ and $V_2 \in N(V_1)$. Repeating this procedure k times we will obtain two paths in $P_k(G)$, $U_k \ldots U_1U$ and $VV_1 \ldots V_k$, where $U_k = x_1 \ldots x_ku_0$ and $V_k = x_1 \ldots x_kv_0$. Then, $U_k = V_k$ because $u_0 = v_0$ and we have a path in $P_k(G)$, the path $U, U_1, \ldots, U_{k-1}, U_k, V_{k-1}, \ldots, V_1, V$, joining U and V. Clearly, the length of that path is $2k$. □

The above lemma can be extended to two vertices in $P_k(G)$ whose corresponding paths in G share a vertex, which is not necessarily and endvertex. We are going to prove it, but to simplify the writing we first introduce some notation.

Let P be the path $a_0, a_1, \ldots, a_r$ in G. If $r \geq k$ and no two vertices at distance smaller than or equal to k to P coincide, there exist vertices $a_0a_1 \ldots a_k$ and $a_{r-k}a_{r-k+1} \ldots a_r$ in $P_k(G)$. Moreover, the path P induces a path in $P_k(G)$ between them, which is going to be denoted as $I_k(P)$ or equivalently, $I_k(a_0, a_1, \ldots, a_r)$. Note that $I_k(P)$ has length $r - k$.

Lemma 3.2. *Let k be a positive integer and let G be a graph with minimum degree $\delta \geq 2$, girth $g \geq k+1$ and such that $P_k(G)$ is connected. If U and V are two vertices in $P_k(G)$ determined by paths in G which share a vertex, then there is a path joining U and V. Moreover, such path has length at most $2k$.*

Proof. Let the vertices U and V be determined by the paths $U = u_0u_1 \ldots u_k$ and $V = v_0v_1 \ldots v_k$ in G. If the paths $u_0, u_1, \ldots, u_k$ and $v_0, v_1, \ldots, v_k$ share an endvertex it suffices to apply Lemma 3.1. If not, there exist vertices u_s and v_t such that $u_s = v_t$ and $\{u_0, \ldots, u_{s-1}\} \cap \{v_0, \ldots, v_{t-1}\} = \emptyset$. Without loss of generality we can assume $s \geq t$. Then, proceeding as in the proof of Lemma 3.1, since $\delta \geq 2$ and $g \geq k+1$ it is possible to construct a path $x_k, \ldots, x_{s+1}, u_0$ which gives rise to the paths $I_k(x_k, \ldots, x_{s+1}, u_0, \ldots, u_k)$ and $I_k(x_k, \ldots, x_{s+1}, u_0, \ldots, u_s, \ldots, v_{t+1}, \ldots, v_k)$ in $P_k(G)$. The union of those two paths determines a path in $P_k(G)$ joining U and the vertex $u_{s-t} \ldots u_{s-1}v_t \ldots v_k$. At the same time, the vertex $u_{s-t} \ldots u_{s-1}v_t \ldots v_k$ is connected to V. Indeed, if we proceed as in the proof of Lemma 3.1, since $\delta \geq 2$ and $g \geq k+1$ we can find a path $v_k, y_0, \ldots, y_{t-1}$ in G from which arise the paths $I_k(u_{s-t} \ldots u_{s-1}v_t \ldots v_k, y_0, \ldots, y_{t-1})$ and

$I_k(v_0 \ldots v_k, y_0 \ldots y_{t-1})$ in $P_k(G)$. Thus, the union of those paths connects $u_{s-t} \ldots u_{s-1} v_t \ldots v_k$ with V. As a consequence, there is path joining U and V obtained from the union of the previous paths. Furthermore, the lengths of the four original paths used to connect U and V are respectively $k-s$, $k-t$, t and t, so their union has length $2k-s+t$ and since $s \geq t$, it is at most $2k$. □

Lemma 3.3. *Let k be a positive integer and let G be a connected graph with minimum degree $\delta \geq 2$, girth $g \geq k+1$ and such that $P_k(G)$ is connected. If U and V are two vertices in $P_k(G)$ whose corresponding paths in G do not share any vertex, then there is a path of length at most $2k + D(G)$ joining U and V.*

Proof. Let the vertices U and V be determined by the paths $U = u_0 u_1 \ldots u_k$ and $V = v_0 v_1 \ldots v_k$ in G. Let us assume that the shortest path between $\{u_0, \ldots, u_k\}$ and $\{v_0, \ldots, v_k\}$ is the shortest path between the vertices u_s and v_t, $u_s = z_0, z_1, \ldots, z_d = v_t$. Note that because of this choice, $\{u_0, \ldots, u_k\} \cap \{z_1, \ldots, z_{d-1}\} = \emptyset$ and $\{v_0, \ldots, v_k\} \cap \{z_1, \ldots, z_{d-1}\} = \emptyset$. Since $\delta \geq 2$ and $g \geq k+1$ there exist paths $x_k, \ldots, x_{s+1}, u_0$ and $v_k, y_0, \ldots, y_{t-1}$, in such a way that there are paths $I_k(x_k, \ldots, x_{s+1}, u_0, \ldots, u_k)$, $I_k(x_k, \ldots, x_{s+1}, u_0, \ldots, u_s, z_1, \ldots, z_{d-1}, v_t, \ldots, v_k, y_0, \ldots, y_{t-1})$ and $I_k(v_0, \ldots, v_k, y_0, \ldots, y_{t-1})$. The union of those three paths forms a path joining U and V. Besides, the lengths of those three paths are respectively $k-s$, $d+k-t$ and t. Therefore, the total length will be $2k+d-s$. Since $s \geq 0$ and $d \leq D(G)$, we conclude that the length of the path between U and V is at most $2k + D(G)$. □

As a direct consequence of the previous lemmas we can obtain the following theorem regarding the diameter.

Theorem 3.4. *Let k be a positive integer and let G be a graph with minimum degree $\delta \geq 2$, girth $g \geq k+1$ and such that $P_k(G)$ is connected. Then, $D(P_k(G)) \leq D(G) + 2k$.*

The previous theorem complement the results from Knor and Niepel [6] and Belan and Jurica [2]. Moreover, they improve the upper bounds presented by Belan and Jurica for $2 \leq k \leq 4$.

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