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Fault Tolerance in Digraphs of Small Diameter

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Abstract

The length and wide of containers between any pair of vertices of a digraph give a good measure of the fault tolerance of an interconnection network. Indeed, many parameters related to the fault tolerance can be calculated from them, as the connectivity, wide-diameter and fault-diameter. In this paper we give containers and find the values of the connectivities, wide-diameters and fault-diameters for a given class of digraphs of small diameter.

Key words: containers, wide-diameter, fault-diameter

1. Introduction

Interconnection networks are usually modeled by graphs, where the switching elements or processors are represented by the vertices, while the communication links are represented by edges (if they are bidirectional) or arcs (if they are unidirectional). Some concepts on graphs appear to be especially useful in order to analyze the efficiency and the reliability of an interconnection network model. For example, the wide-distance, wide-diameter and fault-diameter are a good measure of the fault tolerance capability of a network [8, 9, 10].

Some upper bounds for the wide-diameter and fault-diameter in de Bruijn and Kautz digraphs have been studied in [2, 11]. In [13] similar results were presented for the bipartite digraphs $BD(d, d^{D-3} + d^{D-1})$ [6]. More generally, in [3] bounds were presented for the wide-diameter and fault-diameter in Bruijn and Kautz generalized cycles [7]. The previous results arise from particular topological properties of each family of networks.

2. Definitions

Let $G = (V, E)$ denote a simple digraph with no loops or parallel edges. The vertex connectivity of a digraph G is $\kappa(G)$, defined as the minimum number of vertices whose deletion disconnects G. Analogously the arc connectivity $\lambda(G)$ id the minimum number of arcs to be removed in order to disconnect the graphs. The $(w-1)$-*vertex-fault-diameter*, $D_w(G)$, of a graph G is the maximum of the diameters of the digraphs obtained by removing at most $w-1$ vertices from G. The $(w-1)$-*arc-fault-diameter*, $D'_w(G)$, is defined analogously.

Let x and y be two vertices of a digraph G. Two paths from x to y are said to be *vertex-disjoint* or simply *disjoint* if they do not have any internal vertex in common. A *container* from a vertex x to another vertex y is a set $C(x, y)$ of disjoint paths from x to y. The *width* $w(C(x, y)$ of a container $C(x, y)$ is the number of disjoint paths that it has, and its *length* $l(C(c, y))$, is the maximum length of its paths. For an integer w, $0 \leq w \leq \kappa(G)$, the *w-wide-distance* from x to y, $d_w(x, y)$, is the minimum length of all containers of width w from x to y. Finally, the *w-wide-diameter* of the digraph G, $d_w(G)$, is the maximum w-wide-distance among all pairs of different vertices in G.

In general, the following relations hold: $d_w(G) \geq D_w(G)$ and $d_w(G) \geq D'_w(G)$. Also there exist some relations between these two parameters, the connectivity and the diameter. From the definition, $D_1(G)$ and $D'_1(G)$ coincide with the diameter of G. If $\kappa = \kappa(G)$ there is a container of width κ between every pair of distinct nodes. Clearly, $D_w(G) \leq D_{w+1}(G)$ and $D'_w(G) \leq D'_{w+1}(G)$. In particular, since $D(G) = \infty$ if G is not connected, the vertex-connectivity $\kappa = \kappa(G)$ and the arc-connectivity $\lambda = \lambda(G)$ are, respectively, the minimum values of w satisfying $D_{w+1}(G) = \infty$ and $D'_{w+1}(G) = \infty$.

For a path P in G, $|P|$ will denote its length. If x and y are two vertices in P, the subpath of P from x to y will be denoted as $P(x, y)$.

For more information on graph concepts the reader is referred to [1].

3. Containers

We start by introducing a new parameter on digraphs that allows us to define the class of graphs we are going to work on.

Definition 3.1 *Let G be a simple digraph with diameter D, we define $r = r(G)$ as the greatest integer $1 \leq r \leq D$, such that for any pair of vertices x, y hold:*

(1) if $d(x, y) < r$, there is only one shortest path from x to y, and every other path has length greater than r;

(2) if $d(x, y) = r$, there is only one shortest path from x to y.

This parameter is closely related to the parameters ℓ_0 [5], ℓ_1^* [12] and $\ell_{\pi,r}$ [4] which have been shown to be useful to study the fault tolerance of iterated line digraphs.

As an example, note that for C_n, the cycle of length n, since there is only one shortest path between any two vertices, we have $r(C_n) = n-1$. Analogously for P_n, the path of length n, $r(P_n) = n$. On the other hand, for the complete digraph on n vertices K_n, the shortest path is unique only between adjacent vertices, so $r(K_n) = 1$.

The following two theorems provide a method to construct containers between any two vertices, whose length exceed the diameter in at most two units.

Theorem 3.1 *Let $G = (V, E)$ be a simple digraph with minimum degree d, diameter D and parameter $r = r(G)$. If $D \leq 2r - 1$, there exists a container of wide d and length at most $D + 2$ between every pair of non-adjacent vertices.*

Proof: Let u and v be a pair of non-adjacent vertices of G. Since d is the minimum degree, there exist sets of vertices $\{u_1, u_2, \dots, u_d\}$ and $\{v_1, v_2, \dots, v_d\}$, adjacent from u and to v, respectively. Now, among all the paths from $\{u_1, u_2, \dots, u_d\}$ to $\{v_1, v_2, \dots, v_d\}$, we consider a shortest one, let us say, P_1. For simplicity, we assume that its first vertex after u is u_1, and the last one before v is v_1. Then, we choose a shortest path from $\{u_2, \dots, u_d\}$ to $\{v_2, \dots, v_d\}$, let us say P_2, and again we suppose that u_2 and v_2 are the first vertex after u and the last one before v, respectively. Repeating this procedure, we finally have paths $P_1, P_2, \dots, P_d$, where for each $i = 1, \dots, d$, P_i is a shortest path from $\{u_i, \dots, u_d\}$ to $\{v_i, \dots, v_d\}$. We are going to prove that they are disjoint when the condition $D \leq 2r - 1$ holds.

Suppose that two paths P_i and P_j $(i < j)$ have a common vertex, let us say w. By the construction, $d(u_i, w) + d(w, v_i) \leq D \leq 2r - 1$. This implies that at least one of the distances, $d(u_i, w)$ or $d(w, v_i)$ is at most $r - 1$. Without loss of generality, suppose that $d(u_i, w) \leq r - 1$. Then, there is a path from u to w of length at most r, and since the paths $u, u_i, \dots w$ and $u, u_j, \dots w$ are different, by the definition of the parameter r it must be $d(u_j, w) \geq r$. Now, if $d(w, v_i) \leq r-1$, reasoning as before we conclude that $d(w, v_j) \geq r$, so $d(u_j, w) + d(w, v_j) \geq 2r$, which contradicts $d(u_j, w) + d(w, v_j) \leq D \leq 2r-1$. If not, $d(w, v_i) \geq r$ and then, because of the construction of P_i and P_j, since $i < j$, it must be $d(w, v_j) \geq d(w, v_i)$ so again we have $d(u_j, w) + d(w, v_j) \geq 2r$, which is a contradiction.

Note that if $w = u_i$, then P_j connects u to u_i. Since $d(u, u_i) = 1 \leq r$, then $|P_j(u, u_i)| > r$ and $|P_j(u_i, v_j)| < r$ because $|P_j| \leq D$. This is a contradiction due to the construction of the paths. Indeed, $|P_i(u_i, v)| - 1 \leq |P_j(u_i, v)| - 1$, so $|P_i(u_i, v)| \leq |P_j(u_i, v)| \leq r$ so it must be $P_i(u_i, v) = P_j(u_i, v)$ which is not possible because $v_i \neq v_j$. As a result, $u \neq u_i$, and analogously it can be proved that $u \neq u_j$, $v \neq v_i$ and $v \neq v_j$.

Since P_i is a shortest path between u_i and v_i for all $i = 1, \ldots, d$, length of the container is upper bounded by $D+2$. ■

There is also an arc version of the previous theorem.

Theorem 3.2 *Let $G = (V, E)$ be a simple digraph with minimum degree d, diameter D and parameter $r = r(G)$. If $D \leq 2r$, there exist d arc-disjoint paths of length at most $D+2$ between every pair of different vertices.*

Proof: Let u and v be a pair of different vertices of G. Since d is the minimum degree, if u is not adjacent to v, there exist vertex sets $\{u_1, u_2, \ldots, u_d\}$ and $\{v_1, v_2, \ldots, v_d\}$, adjacent from u and to v, respectively. Proceeding as in the proof of Theorem 3.1 we construct paths $P_1, P_2, \ldots, P_d$ from $\{u_1, u_2, \ldots, u_d\}$ to $\{v_1, v_2, \ldots, v_d\}$. If uv is an arc, we choose P_1 to be such arc. Then, we consider the sets of vertices $\{u_2, \ldots, u_d\}$ and $\{v_2, \ldots, v_d\}$, adjacent from u and to v to construct paths $P_2, \ldots, P_d$ as before. We are going to prove that the paths $P_1, \ldots, P_d$ are arc-disjoint under the assumption $D \leq 2r$. Suppose that two paths P_i and P_j $(i < j)$ have a common arc, let us say wz. By the construction $d(u_i, w) + 1 + d(z, v_i) \leq D \leq 2r$, which implies that at least one of the distances, $d(u_i, w)$ or $d(z, v_i)$, is at most $r-1$. Suppose that $d(u_i, w) \leq r-1$. Then, there is a path from u to w of length at most r, so it must be $d(u_j, w) \geq r$. Now, if $d(z, v_i) \leq r-1$, as before we obtain $d(z, v_j) + 1 > r$ and therefore $d(u_j, w) + d(z, v_j) > 2r$, which contradicts $d(u_j, w) + 1 + d(z, v_j) \leq D \leq 2r$. If not, then $d(z, v_i) \geq r$ and since $i < j$, we have $d(z, v_j) > r$. As a consequence, $d(u_j, w) + d(z, v_j) \geq 2r$, which is also a contradiction. In the same way as it was done in the proof of Theorem 3.1, it is possible to bound the length of the paths $P_1, \ldots, P_d$ by $D+2$. ■

4. Fault Tolerance

This section is devoted to use the previous results about containers to obtain values of several parameters concerning with the fault tolerance. These are the wide-diameters, connectivities and fault-diameters.

The next two corollaries of Theorems 3.1 and 3.2 respectively, give us the exact values for the wide-diameters.

Corollary 4.3 *Let $G = (V, E)$ be a simple digraph with minimum degree d, diameter D and parameter $r = r(G)$. If $D \leq 2r-1$,*

$$d_w(G) = \begin{cases} D+1, & w = 2, \ldots d-1; \\ D+2, & w = d. \end{cases} \blacksquare$$

Corollary 4.4 *Let $G = (V, E)$ be a simple digraph with minimum degree d, diameter D and parameter $r = r(G)$. If $D \leq 2r$,*

$$d'_w(G) = \begin{cases} D+1, & w = 2, \ldots d-1; \\ D+2, & w = d. \end{cases} \blacksquare$$

The vertex connectivity $\kappa(G)$ is the minimum value of w satisfying $D_{w+1}(G) = \infty$. Since $d_{w+1}(G) \leq D_{w+1}(G)$, we have the following corollary of Theorem 3.1.

Corollary 4.5 *Let $G = (V, E)$ be a simple digraph with minimum degree d, diameter D and parameter $r = r(G)$. Then,*

$$\kappa(G) = d, \text{ if } D \leq 2r - 1. \blacksquare$$

Analogously, since $d'_{w+1}(G) \leq D'_{w+1}(G)$, we have the next corollary of Theorem 3.2 for the arc connectivity $\lambda(G)$.

Corollary 4.6 *Let $G = (V, E)$ be a simple digraph with minimum degree d, diameter D and parameter $r = r(G)$. Then,*

$$\lambda(G) = d, \text{ if } D \leq 2r. \blacksquare$$

For the fault-diameters, the following upper bounds arise from Theorems 3.1 and 3.2, respectively.

Corollary 4.7 *Let $G = (V, E)$ be a simple digraph with minimum degree d, diameter D and parameter $r = r(G)$. If $D \leq 2r - 1$,*

$$D_w(G) \leq \begin{cases} D+1, & w = 2, \ldots d-1; \\ D+2, & w = d. \blacksquare \end{cases}$$

Corollary 4.8 *Let $G = (V, E)$ be a simple digraph with minimum degree d, diameter D and parameter $r = r(G)$. If $D \leq 2r$,*

$$D'_w(G) \leq \begin{cases} D+1, & w = 2, \ldots d-1; \\ D+2, & w = d. \blacksquare \end{cases}$$

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