FUNCTIGRAPHS: AN EXTENSION OF PERMUTATION GRAPHS

Andrew Chen, Moorhead, Daniela Ferrero, San Marcos,
Ralucca Gera, Monterey, Eunjeong Yi, Galveston

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Abstract. Let G_1 and G_2 be copies of a graph G, and let $f\colon V(G_1) \to V(G_2)$ be a function. Then a functigraph $C(G,f) = (V,E)$ is a generalization of a permutation graph, where $V = V(G_1) \cup V(G_2)$ and $E = E(G_1) \cup E(G_2) \cup \{uv\colon u \in V(G_1), v \in V(G_2), v = f(u)\}$. In this paper, we study colorability and planarity of functigraphs.

Keywords: permutation graph, generalized Petersen graph, functigraph

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1. Introduction and definitions

Throughout this paper, $G = (V(G), E(G))$ stands for a non-empty, simple and connected graph with order $|V(G)|$ and size $|E(G)|$. For a given graph G and $S \subseteq V(G)$, we denote by $\langle S\rangle$ the subgraph of G induced by S. The *distance*, $d(u,v)$, between two vertices u and v in G is the number of edges on a shortest path between u and v in G.

A graph G is *planar* if it can be embedded in the plane. A connected graph, with order at least 3, is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the exterior region [2]. A vertex v in a connected graph G is a *cut-vertex* of G if $G - v$ is disconnected. A *(vertex) proper coloring* of G is an assignment of labels, traditionally called colors, to the vertices of a graph such that no two adjacent vertices share the same color. A coloring using at most $k \geqslant 1$ colors is called a *(proper) k-coloring* and it is equivalent to the problem of partitioning the vertex set into k or fewer independent sets. The least number of colors needed to properly color a graph G is the chromatic number $\chi(G)$.

Following Chartrand and Harary (see p. 434 of [2]), a *permutation graph* P_α consists of two identical disjoint copies of a labeled graph G, say G_1 and G_2, along with $|V(G)| = n$ additional edges joining $V(G_1)$ and $V(G_2)$ according to a given permutation α on $\{1, 2, \ldots, n\}$. As noted by the authors, the graph $P_\alpha(G)$ depends not only on the choice of the permutation α but on the particular labeling of G as well.

For additional graph theory terminology we refer to [3]. We now introduce the study of a functigraph. We first recall that a *function graph* was independently introduced and was studied by Stephen Hedetniemi in [4] which overlaps our definition in a special case.

Definition 1.1. Let G_1 and G_2 be two copies of a graph G with disjoint vertex sets $V(G_1)$ and $V(G_2)$, and let f be a function from $V(G_1)$ to $V(G_2)$. We define the functigraph $C(G, f)$ to be the graph that has the vertex set

$$V(C(G, f)) = V(G_1) \cup V(G_2),$$

and the edge set

$$E(C(G, f)) = E(G_1) \cup E(G_2) \cup \{uv\colon\ u \in V(G_1), v \in V(G_2), v = f(u)\}.$$

We refer to $V(G_1)$ as the *domain* of the function f, to $V(G_2)$ as the *codomain* of f, and to $f(V(G_1))$ as the *range* of f.

Note that we use the notation $C(G, f)$ to refer to functigraphs for idiosyncratic reasons. A mnemonic is that the C reminds us that we have two Copies of G with function f mapping between them.

Observe that since G_1 and G_2 are copies of the same graph, the function f could be invertible. If so, then $C(G, f)$ is isomorphic to $C(G, f^{-1})$, which is a permutation graph. Indeed, functigraphs generalize permutation graphs [2]. That is, for any graph G, if f is a permutation on $V(G)$, then the functigraph $C(G, f)$ and the permutation graph $P_\alpha(G)$ coincide. The class of permutation graphs and thus the functigraphs include several interesting families of graphs such as

(1) The prisms $C(C_n, f)$ where $f\colon\ V(G_1) \to V(G_2)$ is defined by

$$f(x) = \begin{cases} x + k & \text{if } 1 \leqslant x + k \leqslant n, \\ x + k - n & \text{if } x + k > n. \end{cases}$$

for $0 \leqslant k \leqslant n - 1$ (see (A) of Figure 1 when $k = 1$).

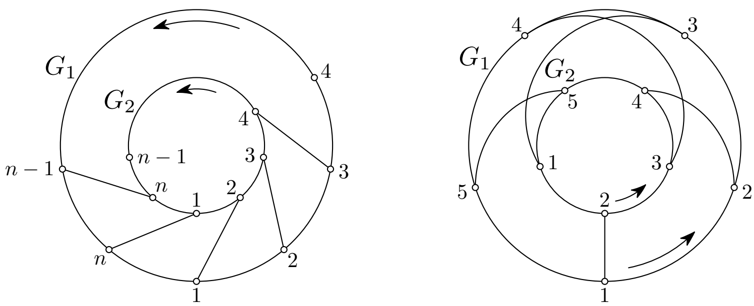
(2) The Petersen graph $C(C_5, f)$ where $f\colon\ V(G_1) \to V(G_2)$ is defined by

$$f(x) = \begin{cases} 2x & \text{if } x = 1, 2, \\ 2x - 5 & \text{if } x = 3, 4, 5. \end{cases}$$

See (B) of Figure 1.

(3) The hypercubes Q_n which are $C(Q_{n-1}, f(x) = x)$, $n \geqslant 1$.

(4) $G \times K_2$, where G is any connected graph, in particular the ladder graphs $P_n \times P_2$ which are $C(P_n, f(x) = x)$.



(A) a prism (B) Petersen graph

Figure 1. A prism and the Petersen graph

In this paper, we characterize the proper colorability and planarity in $C(G, f)$ when G is a cycle. In addition, we study colorability and planarity in $C(G, f)$ for an arbitrary graph G.

2. Colorability of functigraphs

In this section we investigate the (proper) colorability of functigraphs. Clearly, $\chi(C(G, f)) \geqslant \chi(G)$, for all graphs G. Chartrand and Frechen [1] proved that, for every graph G and every permutation graph $P_\alpha(G)$ of G, $\chi(G) \leqslant \chi(P_\alpha(G)) \leqslant \lceil \frac{4}{3}\chi(G) \rceil$.

In this section, we generalize the result by Chartrand and Frechen for an arbitrary function $f\colon V(G_1) \to V(G_2)$. We begin our study by letting $G = C_n$ and then we proceed to arbitrary graphs. Let $C_n\colon v_1, v_2, \ldots, v_n, v_1$ be a cycle of length n. For simplicity, we refer to a vertex by the index i of its label v_i $(1 \leqslant i \leqslant n)$.

Proposition 2.1. *Let $G = C_n$ for n even. Then $2 \leqslant \chi(C(G, f)) \leqslant 3$. Moreover, for all $i, j \in V(G_1)$, $d(i, j)$ and $d(f(i), f(j))$ have the same parity if and only if $\chi(C(G, f)) = 2$.*

Proof. Let $G = C_n$ be an n-cycle for n even. Then $\chi(G_1) = \chi(G_2) = 2$, so the lower bound follows.

Now we consider the upper bound. Since G_2 is an even cycle, there is a 2-coloring of G_2, say c, using colors 1 and 2. Let $S_t = \{x \in V(G_1)\colon c(f(x)) = t\}$ for $t = 1, 2$. We next present a coloring of G, where vertices in S_1 are only colored 2 or 3, and vertices in S_2 are colored 1 or 3. Let every other vertex in G_1 be colored 3, say these vertices form the set S. For $x \in S_1 - S$ let the color of x be 2, and if $x \in S_2 - S$ let the color of x be 1. Thus $\chi(C(G,f)) = 3$, giving the upper bound.

To see the characterization, note that if for all $i, j \in V(G_1)$, $d(i,j)$ and $d(f(i), f(j))$ have the same parity, then all cycles in $C(C_n, f)$ are even. This is the case if and only if $C(G,f)$ is bipartite, i.e. $\chi(C(G,f)) = 2$. For the converse, if there exist $i, j \in V(G_1)$ with $d(i,j)$ and $d(f(i), f(j))$ having different parities, then $C(G,f)$ contains an odd cycle, and so $\chi(C(G,f)) \geqslant 3$. □

For an odd cycle C_n, we have the following:

Proposition 2.2. *Let $G = C_n$ for n odd. Then $3 \leqslant \chi(C(G,f)) \leqslant 4$. Moreover, $\chi(C(G,f)) = 4$ if and only if f is a constant function.*

Proof. Since $C(G,f)$ contains an odd cycle, $\chi(C(G,f)) \geqslant 3$. To obtain the upper bound, we show that a coloring with at most 4 colors can be found, by considering two cases.

Case 1. f is a constant function. Say $f\colon V(G_1) \to V(G_2)$ given by $f(x) = a$ for all $x \in V(G_1)$ and for some $a \in V(G_2)$ with $1 \leqslant a \leqslant n$. Define a coloring $c_1\colon V(C(G,f)) \to \{1,2,3,4\}$ by 3-coloring G_1 with colors $1, 2$, and 3, color the vertex labeled a by 4, and properly color the rest of G_2 with two colors 1 and 2. Thus $\chi(C(G,f)) \leqslant 4$.

Case 2. f is not a constant function. Since G_2 is an odd cycle, there exists a 3-coloring of $V(G_2)$, say c, using colors 1, 2, and 3. Without loss of generality, color G_2 so that color 3 is only used once, and it is used on a vertex of G_2 of minimum degree in $C(G,f)$. Let $S_t = \{x \in V(G_1)\colon c(f(x)) = t\}$ for $t = 1, 2, 3$. Thus $|S_3| \in \{0, 1\}$. We consider two subcases.

Subcase 2.1. $|S_3| = 0$. First we assume that either $S_1 = \emptyset$ or $S_2 = \emptyset$, say the former. Then $V(G_1) = S_2$ and one can properly color G_1 with colors 1, 3, and 4. Thus $\chi(C(G,f)) \leqslant 4$. Next we assume that $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. Then find two consecutive vertices in G_1, say k and $k+1$ such that $k \in S_1$ and $k+1 \in S_2$, or $k \in S_2$ and $k+1 \in S_1$, say the former. Then complete the 3-coloring c_2 of $C(G,f)$ by coloring every other vertex of the cycle G_1 starting with $k+2$ with color 3, say these vertices form the set S. For $i \in V(G_1) - S$ let the color of i be

$$c_2(i) = \begin{cases} 2 & \text{if } i \in S_1, \\ 1 & \text{if } i \in S_2. \end{cases}$$

And so $\chi(C(G,f)) = 3$ in this case.

Subcase 2.2. $|S_3| = 1$. Then let $V(G_1) \cap S_3 = \{k\}$. Then either $k+1 \in S_1$ or $k+1 \in S_2$, say the former. Now define a coloring $c_2\colon V(C(G,f)) \to \{1,2,3\}$ by letting every other vertex in G_1 be colored 3 starting with $k+1$, say these vertices form the set S. For $x \in S_1 - S$ let the color of x be 2, if $x \in S_2 - S$ let the color of x be 1, and the color of k is

$$c_2(k) = \begin{cases} 1 & \text{if } k-1 \in S_1, \\ 2 & \text{if } k-1 \in S_2. \end{cases}$$

And so, $\chi(C(G,f)) = 3$.

We now prove the characterization. Assume that $f(x) = a$ for all $x \in V(G_1)$ and for some a with $1 \leqslant a \leqslant n$. Since the vertex labeled a in G_2 is adjacent to every vertex of the odd cycle G_1, we have that $\chi(C(G,f)) \geqslant 4$, and by the proof above we have that $\chi(C(G,f)) \leqslant 4$, thus $\chi(C(G,f)) = 4$.

For the converse, assume to the contrary that f is not a constant function. Then the coloring c_2 above proves that $\chi(C(G,f)) = 3$, a contradiction. □

We now give bounds for the chromatic number of the functigraph $C(G,f)$ in terms of the chromatic number of the graph G, for any graph G and for all functions f. The upper bound of the theorem below is a special case of a result independently proved by Hedetniemi in [4].

Theorem 2.3. *If $\chi(G) = \alpha$, then $\alpha \leqslant \chi(C(G,f)) \leqslant \alpha + \lceil \frac{1}{2}\alpha \rceil$. Both bounds are sharp.*

Proof. Since $C(G,f)$ contains a copy of G, $\chi(C(G,f)) \geqslant \alpha$. For the upper bound let G_1 and G_2 be the two copies of G in $C(G,f)$. Let c^* be a coloring of G_2 with the color classes $1, 2, \ldots, \alpha$ such that $W_i = \{w \in V(G_2)\colon c^*(w) = i\}$ for $1 \leqslant i \leqslant \alpha$ with $V(G_2) = \bigcup_{i=1}^{\alpha} W_i$. And let $S_i = \{v \in V(G_1)\colon c^*(f(v)) = i\}$ for $1 \leqslant i \leqslant \alpha$. Since G_1 is also α-partite, it follows that $V(G_1) = \bigcup_{i=1}^{\alpha} U_i$, where U_i is the independent set corresponding to W_i $(1 \leqslant i \leqslant \alpha)$. We construct the coloring c of $C(G,f)$ by $c\colon V(C(G,f)) \to \{1, 2, \ldots, \lceil \frac{3}{2}\alpha \rceil\}$, where

$$c(v) = \begin{cases} i & \text{if } v \in W_i \ (1 \leqslant i \leqslant \alpha), \\ j + \lceil \frac{1}{2}\alpha \rceil & \text{if } v \in U_j \cap S_j \ (1 \leqslant j \leqslant \lfloor \frac{1}{2}\alpha \rfloor), \\ j & \text{if } v \in U_j - S_j \ (1 \leqslant j \leqslant \lfloor \frac{1}{2}\alpha \rfloor), \\ 2\alpha + 1 - k & \text{if } v \in U_k \ (\lfloor \frac{1}{2}\alpha \rfloor + 1 \leqslant k \leqslant \alpha). \end{cases}$$

And so $\chi(C(G,f)) \leqslant \alpha + \lceil \frac{1}{2}\alpha \rceil$.

To see the sharpness of the lower bound, let G be an a-partite graph with $V(G) = \{V_1, V_2, \ldots, V_a\}$ being a partition of $V(G)$ into independent sets, and f be the identity function. Let G_1 and G_2 be the two copies of G, with $V(G_1) = \{V_1^1, V_2^1, \ldots, V_a^1\}$ and $V(G_2) = \{V_1^2, V_2^2, \ldots, V_a^2\}$ be the corresponding partitions into independent sets of $V(G_1)$ and $V(G_2)$. Then define a coloring $c\colon V(C(G,f)) \to \{1, 2, \ldots, a\}$ given by

$$c(v) = \begin{cases} i & \text{if } v \in V_i^1 \ (1 \leqslant i \leqslant a), \\ j+1 & \text{if } v \in V_j^2 \ (1 \leqslant j \leqslant a-1), \\ 1 & \text{if } v \in V_a^2. \end{cases}$$

Thus $\chi(C(G,f)) = \chi(G) = a$.

To see the sharpness of the upper bound, let (1) G be a complete a-partite graph $K_{a,a,\ldots,a}$, (2) G_1 be a copy of G with the partition of $V(G_1)$ into independent classes as $V(G_1) = \bigcup_{i=1}^{a} U_i$, where $U_i = \{u_{i1}, u_{i2}, \ldots, u_{ia}\}$, (3) G_2 be another copy of G with the partition of $V(G_2)$ into independent classes as $V(G_2) = \bigcup_{j=1}^{a} W_j$, where $W_j = \{w_{j1}, w_{j2}, \ldots, w_{ja}\}$, and (4) $f(u_{ij}) = w_{jj}$, for all i, j, $1 \leqslant i, j \leqslant a$.

Then define the coloring $c\colon V(C(G,f)) \to \{1, 2, \ldots, \lceil \frac{3}{2}a \rceil\}$ given by

$$c(v) = \begin{cases} j & \text{if } v \in W_j \ (1 \leqslant j \leqslant a), \\ i + \lceil \frac{1}{2}a \rceil & \text{if } v = u_{ij} \ (i = j, 1 \leqslant i \leqslant \lfloor \frac{1}{2}a \rfloor), \\ i & \text{if } v = u_{ij} \ (i \neq j, 1 \leqslant i \leqslant \lfloor \frac{1}{2}a \rfloor), \\ 2a + 1 - i & \text{if } v \in U_i \ (\lfloor \frac{1}{2}a \rfloor + 1 \leqslant i \leqslant a). \end{cases}$$

Thus $\chi(C(G,f)) \leqslant \lceil \frac{3}{2}a \rceil$. We claim that $\chi(C(G,f)) = \lceil \frac{3}{2}a \rceil$. Assume, to the contrary, that $\chi(C(G,f)) = l < \lceil \frac{3}{2}a \rceil$. Then $l < \frac{3}{2}a$. Since we have that $\langle \{w_{11}, w_{22}, \ldots, w_{aa}\} \rangle \cong K_a$, we then obtain $\chi(\langle \{w_{11}, w_{22}, \ldots, w_{aa}\} \rangle) = a$. Without loss of generality, we assign $c(w_{jj}) = j$ for $1 \leqslant j \leqslant a$. Suppose that r of the sets U_i have the same color assigned to each vertex of set. Since each vertex of U_i $(1 \leqslant i \leqslant a)$ is adjacent to a different vertex of the clique $\{w_{11}, w_{22}, \ldots, w_{aa}\}$, it follows that the r colors must be distinct from the a colors already used. Then $\chi(C(G,f)) \geqslant a + r$. Since at least two colors are used for each of the remaining $a - r$ sets of U_i, $r + 2(a - r) \leqslant l$, and so $r \geqslant 2a - l$. Therefore, $\chi(C(G,f)) \geqslant a + r \geqslant a + 2a - l = 3a - l > 3a - \frac{3}{2}a = \frac{3}{2}a > l = \chi(C(G,f))$, which is a contradiction. □

Below is an example that shows the construction of the coloring of $C(G,f)$ for $G = K_{3,3,3}$ for the sharpness of the the upper bound above.

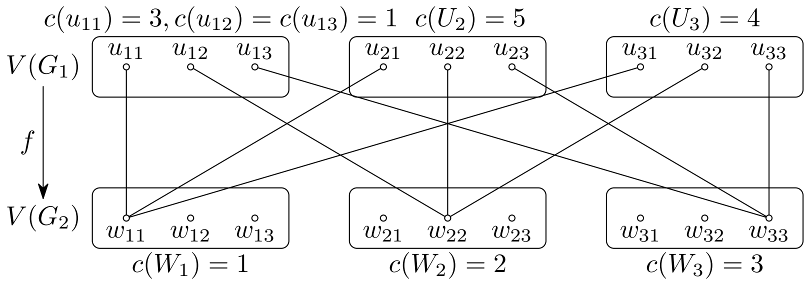


Figure 2. An example of $\chi(C(K_{3,3,3}, f)) = 5$

2. Planarity of functigraphs

Chartrand and Harary [2] proved a result analogous to Kuratowski's theorem for outerplanar graphs: a connected graph G is outerplanar if and only if it contains no subgraph which is homeomorphic to K_4 or $K_{2,3}$. Further, Chartrand and Harary characterized planar permutation graphs that contain no cut-vertices. The result states that the permutation graph $P_\alpha(G)$ of a nonseparable graph G is planar if and only if G is outerplanar and α is *dihedral*. (See Ch. 4 of [5] for the dihedral groups.)

We begin this section with a characterization of planar functigraphs $C(C_n, f)$. We further characterize planar functigraphs $C(G, f)$ for an arbitrary graph G, and thus generalize the results obtained by Chartrand and Harary on permutation graphs to functigraphs.

We first consider $G = C_n$. Let $C_n\colon v_1, v_2, \ldots, v_n, v_1$ be a cycle of length n. For simplicity, we refer to each vertex of the cycle by its index i of its label v_i $(1 \leqslant i \leqslant n)$. Let G_1 and G_2 be already embedded in the plane, with both labeled counterclockwise as in (A) of Figure 3. Let $f\colon V(G_1) \to V(G_2)$ be a function and let $\sigma\colon V(G_2) \to V(G_2')$ be an identity function, where G_2' is a copy of G_2 with clockwise orientation. Then the composition $g := \sigma \circ f$ maps (B) of Figure 3 to (A) of Figure 3. Thus we only need to consider when $V(G_1)$ and $V(G_2)$ are labeled counterclockwise as in (A) of Figure 3.

For the rest of the paper, the function f such that $f(i) = a_i$ for all $i = 1, 2, \ldots, n$ will be denoted by $f = (a_1, a_2, \ldots, a_n)$. Note that if all a_i's are distinct and $\bigcup\limits_{i=1}^{n} \{a_i\} = \bigcup\limits_{j=1}^{n} \{j\}$, then $f\colon V(G_1) \to V(G_2)$ is a bijection and thus $C(G, f)$ is a permutation graph.

Example. Let $G = C_5$ be a cycle of length 5. Let G_1 and G_2 be copies of G with labelings of vertices assigned as in Figure 4. Then $f = (2, 2, 3, 3, 1)$ means

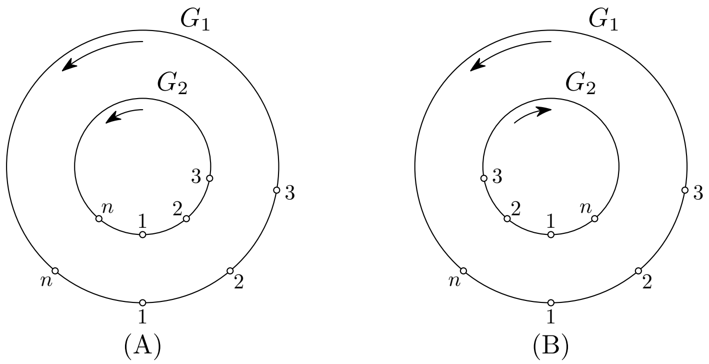


Figure 3. Labelings of $V(G_1)$ and $V(G_2)$

that $f(1) = 2$, $f(2) = 2$, $f(3) = 3$, $f(4) = 3$, and $f(5) = 1$. Note that $C(G, f)$ is isomorphic to $C(G, \widetilde{f})$ for $\widetilde{f} = (1, 1, 2, 2, 5)$.

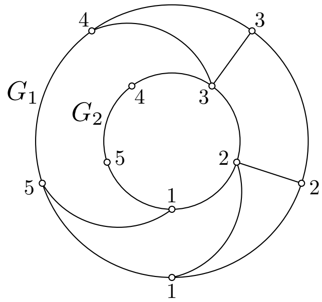


Figure 4. $f_1\colon V(G_1) \to V(G_2)$ as in Example

Notice that, by allowing relabeling, we can assume that there exists a vertex in G_1 that is mapped to vertex n in G_2. If $f(n) = a_n \neq n$ then there exists an automorphism $\overline{\sigma}\colon i \to i + n - a_n$ such that $\overline{\sigma} \circ f(n) = n$.

We next characterize planar functigraphs when G is a cycle. One can easily check that $C(C_3, f)$ is planar for any function f: there are three distinct (non-isomorphic) cases. Thus, we consider $G = C_n$ for $n \geqslant 4$.

We define a function f of a functigraph $C(C_n, f)$ to be *semi-monotonic*, and we denote it by $f \in SM_n$, if and only if $f = (a_1, a_2, \ldots, a_n)$ satisfies

(1) there exists $k \in V(G_1)$ $(1 \leqslant k \leqslant n)$ such that

$$1 \leqslant a_k \leqslant a_{k+1} \leqslant \ldots \leqslant a_{n-1} \leqslant a_n \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_{k-1} \leqslant n$$

or

(2) there exists $k \in V(G_1)$ $(1 \leqslant k \leqslant n)$ such that

$$1 \leqslant a_k \leqslant a_{k-1} \leqslant \ldots \leqslant a_2 \leqslant a_1 \leqslant a_n \leqslant a_{n-1} \leqslant \ldots \leqslant a_{k+1} \leqslant n.$$

A graph H is called a *subdivision* of a graph G if one or more vertices of degree 2 are inserted into one or more edges of G (see p. 236 of [3]). Now we recall Kuratowski's Theorem: a graph G is planar if and only if G does not contain K_5, $K_{3,3}$, or a subdivision of K_5 or $K_{3,3}$ as a subgraph.

Proposition 3.1. *Let $G = C_n$ be a cycle of length $n \geqslant 4$. Let G_1 and G_2 be copies of G with cyclic labelings. Without loss of generality, we assume that $f(n) = n$. Further, by relabeling the vertices so that two adjacent vertices 1 and n in G_1 get mapped to two different vertices in G_2, we may assume that $f(1) \neq n$ if f is not a constant function. Then $C(G, f)$ is planar if and only if f is semi-monotonic (f could be a constant function).*

Proof. (⇐) It is easy to check.

(⇒) Let $C(G, f)$ be planar for $G = C_n$ with $n \geqslant 4$. Assume, to the contrary, that f is not semi-monotonic. Then, without loss of generality, we may assume that there exist vertices r, s, t, u in G_1 such that $1 \leqslant r < s < t < u \leqslant n$ and $1 \leqslant a_r \leqslant a_t < a_s \leqslant a_u \leqslant n$. We consider three cases.

Case 1. $|f(V(G_1))| = 2$: Note that $C(G, f)$ contains a subdivision of $H_1 \cong K_{3,3}$ (see Figure 5) as a subgraph.

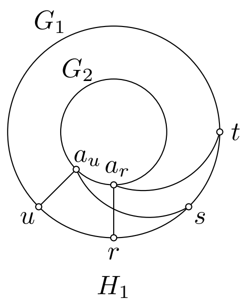
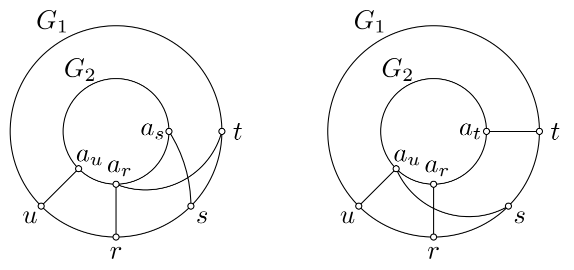


Figure 5

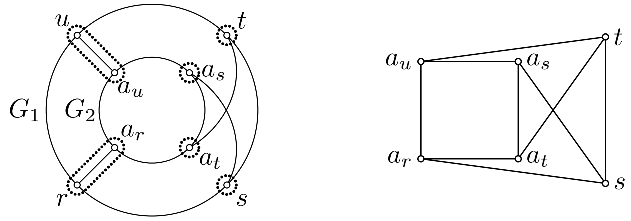
Case 2. $|f(V(G_1))| = 3$: Notice that $C(G, f)$ contains a subdivision of either H_1, H_2, or H_2^* as a subgraph. Moreover, H_2 also contains a subdivision of $K_{3,3}$ as a subgraph, where the two bipartite sets are $\{r, t, a_s\}$ and $\{s, u, a_r\}$. Also note that H_2^* contains a subdivision of $K_{3,3}$ as a subgraph, where the two bipartite sets are $\{r, t, a_u\}$ and $\{s, u, a_t\}$ (see Figure 6).



H_2 H_2^*

Figure 6

C a s e 3. $|f(V(G_1))| \geqslant 4$: Notice that $C(G,f)$ contains a subdivision of either H_1, H_2, H_2^*, or H_3 as a subgraph. Moreover H_3 contains a subdivision of $H_3' \cong K_{3,3}$ as a subgraph, where the two bipartite sets are $\{t, a_r, a_s\}$ and $\{s, a_t, a_u\}$ (see Figure 7).



H_3 H_3'

Figure 7

Thus, by Kuratowski's Theorem, $C(G,f)$ is not planar, which is a contradiction to the planarity hypothesis. □

The following is an immediate result of Proposition 3.1. The sufficient direction of the corollary below, for any graph G, was independently proved by Hedetniemi in the study of function graph in [4].

Corollary 3.2. *Let G be a graph without cut-vertices. Then the functigraph $C(G,f)$ is planar if and only if G is outerplanar and f is semi-monotonic (f could be a constant function).*

P r o o f. (⇐) It is easy to check.

(⇒) Assume, to the contrary, that either G is not outerplanar or f is not semi-monotonic. If G is not outerplanar, then there is a vertex $v \in V(G)$ such that v doesn't lie on the exterior region. Then the edge $vf(v)$ will cross one of the edges of G, a contradiction to the planarity hypothesis of $C(G,f)$. Thus G is outerplanar, and we can assume that every nonseparable outerplanar graph G is cyclically labeled. If $f \notin SM_n$, then $C(G,f)$ is not planar, a contradiction. □

R e m a r k. The condition of G having no cut-vertex is necessary. To see this, let $G = P_6$ be a path of length 5, with cut-vertices, as in Figure 8. Define $f\colon V(G_1) \to V(G_2)$ by

$$f(x) = \begin{cases} \dfrac{x+6}{2} & \text{if } x \text{ is even,} \\ \dfrac{x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

Then $C(G, f)$ is planar and G is outerplanar, but $f = (1, 4, 2, 5, 3, 6)$ is not semi-monotonic.

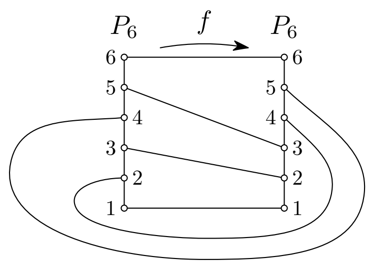


Figure 8. $C(P_6, f)$ for the Remark above

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References

[1] *G. Chartrand, J. B. Frechen*: On the chromatic number of permutation graphs. Proof Tech. Graph Theory, Proc. 2nd Ann Arbor Graph Theory Conf. 1968. 1969, pp. 21–24. zbl

[2] *G. Chartrand, F. Harary*: Planar permutation graphs. Ann. Inst. Henri. Poincaré, Nouv. Sér., Sect. B *3* (1967), 433–438. zbl

[3] *G. Chartrand, P. Zhang*: Introduction to Graph Theory. McGraw-Hill, Kalamazoo, MI, 2004. zbl

[4] *S. Hedetniemi*: On classes of graphs defined by special cutsets of lines. Many Facets of Graph Theory, Proc. Conf. Western Michigan Univ., Kalamazoo/Mi. 1968, Lect. Notes Math. *110* (1969), 171–189. zbl

[5] *T. W. Judson*: Abstract Algebra: theory and applications. Boston, MA: PWS Publishing Company, 1994. zbl

Authors' addresses: *Andrew Chen*, Department of Computer Science and Information Systems, Minnesota State University Moorhead, Moorhead, MN 56563, USA, e-mail: `chenan@mnstate.edu`; *Daniela Ferrero*, Department of Mathematics, Texas State University, San Marcos, TX 78666, USA, e-mail: `dferrero@txstate.edu`; *Ralucca Gera*, Department of Applied Mathematics, Naval Postgraduate School, Monterey, CA 93943, USA, e-mail: `rgera@nps.edu`; *Eunjeong Yi*, Department of General Academics, Texas A&M University at Galveston, Galveston, TX 77553, USA, e-mail: `yie@tamug.edu`.