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New bounds on the diameter vulnerability of iterated line digraphs

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Abstract

Iterated line digraphs have some good properties in relation to the design of interconnection networks. The diameter vulnerability of a digraph is the maximum diameter of the subdigraphs obtained by deleting a fixed number of vertices or arcs. This parameter is related to the fault-tolerance of interconnection networks. In this work, we introduce some new parameters in order to find new bounds for the diameter vulnerability of general iterated line digraphs.

1. Introduction

Interconnection networks are usually modeled by graphs. The switching elements or processors are represented by vertices. The communication links are represented by edges (if they are bidirectional) or arcs (if they are unidirectional). In this work, we are only concerned with directed graphs called digraphs, for short. Some basic concepts we use in this work are recalled in Section 2. For additional concepts, we refer the reader to [1].

Some basic requirements in designing interconnection networks make interesting to find large digraphs with bounded degree, small diameter and easy routing. Because of the compromise between the parameters involved — order, maximum degree and diameter — this situation gives rise to some optimization problems. One of them is the (d, D)-digraph problem, that is, to find digraphs with order as large as possible for fixed values of the maximum degree d and the diameter D. The iteration of the line digraph operator is a powerful technique in order to find large (d, D)-digraphs, that is, digraphs with degree d and diameter D [5]. In fact, the best proposed general solutions to the (d, D)-digraph problem, such as de Bruijn and Kautz digraphs, the bipartite digraphs proposed by Fiol and Yebra [4] and the generalized cycles proposed

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in [6], are iterated line digraphs. Besides, iterated line digraphs have other interesting properties in relation to the design of interconnection networks: for instance, properties related to the connectivity and the existence of Hamilton cycles. Besides, it is possible to construct very simple routing algorithms on iterated line digraphs.

Other requirement for an interconnection network is that the system still works with reasonable efficiency when some elements are faulty. That is, the network must be fault-tolerant. The connectivity, which is the minimum number of vertices or arcs whose deletion disconnects the digraph, and the diameter vulnerability, are parameters related to the fault tolerance of a network.

The diameter vulnerability of a digraph is the maximum of the diameters of the subdigraphs obtained by deleting a given number of vertices or arcs. In this paper, we consider the diameter vulnerability of iterated line digraphs, that is, digraphs defined by the iteration of the line digraph operator. The diameter vulnerability of some interesting particular families of iterated line digraphs has been calculated in [9,11,3]. The diameter-vulnerability of general iterated line digraphs was first considered in [10]. It was proved there that, if an iterated line digraph $L^k(G)$ has maximum connectivity, its diameter vulnerability is bounded by $D(L^k(G)) + C$, where C depends on some properties of the digraph G, but does not depend on the number of iterations.

We introduce new parameters in order to find new bounds on the diameter vulnerability of iterated line digraphs. In general, the bounds we present here are not only tighter than the ones given in [10], but improve them in some other aspects. First of all, the bounds we present here do not need $L^k(G)$ to be maximally connected to be applied. Besides, instead of dealing only with the worst case, that is, when the number of faulty elements is just one unity less than the connectivity, our bounds depend on the number of faulty elements. Finally, the bounds given in [10] can take different values when they are computed for $H_1 = L^k(G)$ or for $H_2 = L^{k'}(L^{k-k'}(G))$, being these two digraphs isomorphic. The bounds we present in this paper avoid this problem.

In the next section, we present the most relevant definitions and the notation we are going to use in the following. In Section 3, we introduce the new parameters and their main properties. We present bounds for the diameter vulnerability in Section 4, and we apply them to some interesting families of digraphs in Section 5.

2. Definitions and notation

A *digraph* $G = (V, A)$ consists of a set of *vertices* V and a set A of ordered pairs of vertices called *arcs*. The arcs in the form (x, x) are called *loops*. The cardinality of V is the *order* of the digraph. If (x, y) is an arc, it is said that x is *adjacent to* y and that y is *adjacent from* x. The set of vertices which are adjacent from (to) a given vertex v is denoted by $\Gamma^+(v)$ ($\Gamma^-(v)$) and its cardinality is the *out-degree* of v, $d^+(v) = |\Gamma^+(v)|$ (*in-degree* of v, $d^-(v) = |\Gamma^-(v)|$). Its minimum value over all vertices is the *minimum out-degree*, δ^+, (*minimum in-degree*, δ^-) of the digraph G. The *minimum degree* of G is $\delta = \min\{\delta^+, \delta^-\}$. The *maximum degree* Δ is defined analogously.

A *path* of length h from a vertex x to a vertex y is a sequence of vertices $x = x_0, x_1, \ldots, x_{h-1}, x_h = y$ where (x_i, x_{i+1}) is an arc. A digraph G is *strongly connected* if for any pair of vertices x, y there exists a path from x to y. The length of a shortest path from x to y is the *distance from x to y*, and it is denoted by $d(x, y)$. Its maximum value over all pairs of vertices is the *diameter* of the digraph, $D(G)$. If G is not strongly connected, $D(G) = \infty$.

Let x and y be two different vertices of a digraph G. If the shortest path from x to y is unique, it will be denoted by $x \to y$. Its first vertex after x will be $v(x \to y)$ and its last one before y will be $v(y \leftarrow x)$. Now, if F is a set of vertices of G and $x \notin F$, $v(x \to F)$ is the set formed by $v(x \to f)$ for every vertex $f \in F$, such that the shortest path from x to f is unique, and $v(x \leftarrow F)$ is defined analogously. Thus, $v(x \to F) \subset \Gamma^+(x)$ and $v(x \leftarrow F) \subset \Gamma^-(x)$.

The *vertex-connectivity* $\kappa = \kappa(G)$ of a digraph $G = (V, A)$ is the minimum cardinality of the subsets of vertices $F \subset V$ such that $G - F$ is not strongly connected or is trivial. The *arc-connectivity* $\lambda = \lambda(G)$ is the minimum number of arcs whose deletion produces a subdigraph of G that is not strongly connected. The *s-vertex-diameter-vulnerability*, $K(s, G)$, of a digraph G is the maximum of the diameters of the subdigraphs obtained by removing at most s vertices from G. The *s-arc-diameter-vulnerability*, $\Lambda(s, G)$, is defined analogously. These parameters are related to the diameter and the connectivity. By the definition, $K(0, G)$ and $\Lambda(0, G)$ coincide with the diameter of G. The connectivities of a non-complete digraph G are the minimum values of s satisfying $K(s, G) = \infty$ and $\Lambda(s, G) = \infty$. That is,

$$D(G) = K(0, G) \leqslant K(1, G) \leqslant \cdots \leqslant K(\kappa - 1, G) < K(\kappa, G) = \infty,$$

$$D(G) = \Lambda(0, G) \leqslant \Lambda(1, G) \leqslant \cdots \leqslant \Lambda(\lambda - 1, G) < \Lambda(\lambda, G) = \infty.$$

In the *line digraph* $L(G)$ of a digraph G each vertex represents an arc of G, that is, $V(L(G)) = \{uv \mid (u, v) \in A(G)\}$. Vertex uv is adjacent to vertex wz if $v = w$, i.e., whenever the arc (u, v) of G is adjacent to the arc (w, z). The iteration of the line digraph operation is a good method to obtain large digraphs with fixed degree and diameter. If G is d-regular with $d > 1$, has diameter D and order n, then $L^k(G)$ is d-regular, has diameter $D + k$ and order $d^k n$, that is, the order increases in an asymptotically optimal way in relation to the diameter. The vertices of the iterated line digraph $L^k(G)$ can be represented by walks of length k in G, that is, sequences of $k + 1$ vertices of G, $x_0 x_1 \ldots x_k$, where (x_i, x_{i+1}) is an arc of G. A vertex $\boldsymbol{x} = x_0 x_1 \ldots x_k$ in $L^k(G)$ is adjacent to $\boldsymbol{y} = x_1 \ldots x_k x_{k+1}$ for all x_{k+1} adjacent from x_k. A path of length h in $L^k(G)$ can be written as a sequence of $k + h + 1$ vertices of G. The vertices of this path are the subsequences of $k + 1$ consecutive vertices of G.

3. Parameters $M_{\pi,r}$ and $\ell_{\pi,r}$

In order to find bounds on the diameter vulnerability of iterated line digraphs, we are going to prove that, under certain conditions, for any given set F of faulty vertices and for any pair of vertices $x, y \notin F$, it is possible to find a path from x to y with bounded length and avoiding F. This path will be in the form $xx_1 \dots x_m \dots y_n \dots y_1 y$, where x_m and y_n are vertices such that $d(x_m, f) + d(f, y_n) > D(G)$ for any $f \in F$. It is constructed recursively in such a way that the distances $d(x_i, f)$ and $d(f, y_i)$ increase with i. We introduce in this section the parameters $M_{\pi,r}$ and $\ell_{\pi,r}$, which are related to the properties of short paths between the vertices of the digraph. These parameters will enable us to find the vertices x_i and y_i together with the values of m and n such that $d(x_m, f) + d(f, y_n) > D(G)$ for any $f \in F$.

Definition 3.1. Let G be a digraph with minimum degree $\delta \geqslant 2$ and diameter $D = D(G)$. Let π be an integer, $0 \leqslant \pi \leqslant \delta - 2$. For any positive integer r, we define $\ell_{\pi,r} = \ell_{\pi,r}(G)$ as the greatest integer, $0 \leqslant \ell_{\pi,r} \leqslant D$, such that for each vertex x there exist sets $\Phi^+_{\pi,r}(x) \subset \Gamma^+(x)$, $\Phi^-_{\pi,r}(x) \subset \Gamma^-(x)$, with $|\Phi^+_{\pi,r}(x)|, |\Phi^-_{\pi,r}(x)| \leqslant \pi$, satisfying:

(1) if $d(x, y) < \ell_{\pi,r}$, there is only one shortest path from x to y and any other path with length smaller than or equal to $d(x, y) + r$ has its first vertex in $\Phi^+_{\pi,r}(x)$ and its last one in $\Phi^-_{\pi,r}(y)$.
(2) if $d(x, y) = \ell_{\pi,r}$, the shortest path from x to y is unique.

This parameter is a generalization of the parameters ℓ_0 [2] and ℓ_1^* [10]. In fact, $\ell_{0,1}(G) = \ell_0(G)$ and $\ell_{1,1}(G) = \ell_1^*(G)$. In next lemma we see how the parameter $\ell_{\pi,r}$ and the sets $\Phi^+_{\pi,r}(x)$ and $\Phi^-_{\pi,r}(x)$ can be used to find a vertex $x_1 \in \Gamma^+(x)$ that avoids the short paths from a vertex x to any other vertex y.

Lemma 3.2. *Let G be a digraph with minimum degree $\delta \geqslant 2$, and $\ell_{\pi,r} = \ell_{\pi,r}(G)$ for an integer π with $0 \leqslant \pi \leqslant \delta - 2$ and a positive integer r. If x, y are two vertices of G, then*

(1) *If $d(x, y) < \ell_{\pi,r}$:*
 (a) *for all $x_1 \in \Gamma^+(x) - \Phi^+_{\pi,r}(x)$ such that $x_1 \neq v(x \to y)$, $d(x_1, y) \geqslant d(x, y) + r$.*
 (b) *for all $y_1 \in \Gamma^-(y) - \Phi^-_{\pi,r}(y)$ such that $y_1 \neq v(y \leftarrow x)$, $d(x, y_1) \geqslant d(x, y) + r$.*
(2) *If $d(x, y) = \ell_{\pi,r}$:*
 (a) *for all $x_1 \in \Gamma^+(x) - \{v(x \to y)\}$, $d(x_1, y) \geqslant \ell_{\pi,r}$.*
 (b) *for all $y_1 \in \Gamma^-(y) - \{v(y \leftarrow x)\}$, $d(x, y_1) \geqslant \ell_{\pi,r}$.*

Proof. If $d(x, y) < \ell_{\pi,r}$, $x_1 \notin \Phi^+_{\pi,r}(x)$ and $x_1 \neq v(x \to y)$, then, the length of any path $xx_1 \dots y$ is greater than $d(x, y) + r$. Therefore, $d(x_1, y) \geqslant d(x, y) + r$. In the same way, $d(x, y_1) \geqslant d(x, y) + r$.

If $d(x, y) = \ell_{\pi,r}$ the shortest path from x to y is unique. A shortest path from $x_1 \neq v(x \to y)$ to y determines a path from x to y. Then, $d(x_1, y) + 1 \geqslant d(x, y) + 1 = \ell_{\pi,r} + 1$. Analogously, $d(x, y_1) \geqslant \ell_{\pi,r}$. □

As a direct consequence of Lemma 3.2, we obtain the following result.

Lemma 3.3. *Let G be a digraph with minimum degree $\delta \geqslant 2$, and let $\ell_{\pi,r} = \ell_{\pi,r}(G)$, where $0 \leqslant \pi \leqslant \delta - 2$. Let F be a set of vertices of G with $1 \leqslant |F| \leqslant \delta - \pi - 1$, and x, y two vertices of G, $x, y \notin F$. Then, for every $m \geqslant 1$,*

(1) *there exists a path $xx_1 \dots x_m$ such that, for all $f \in F$, $d(x_m, f) \geqslant \min\{d(x, f) + rm, \ell_{\pi,r}\}$.*
(2) *there exists a path $y_m \dots y_1 y$ such that, for all $f \in F$, $d(f, y_m) \geqslant \min\{d(f, y) + rm, \ell_{\pi,r}\}$.*

The properties of the parameter $\ell_{\pi,r}$ in relation to the line digraph operator are given by the next two propositions.

Proposition 3.4. *Let G be a digraph with minimum degree $\delta \geqslant 2$. Let π and $r \geqslant 1$ be integers such that $0 \leqslant \pi \leqslant \delta - 2$ and $\ell_{\pi,r}(G) \geqslant 1$. Then,*

(1) *$\ell_{\pi,r}(L(G)) = \ell_{\pi,r}(G) + 1$.*
(2) *For any vertex $\boldsymbol{x} = x_0x_1$ of $L(G)$, $\Phi^+_{\pi,r}(\boldsymbol{x}) = \{x_1 w \mid w \in \Phi^+_{\pi,r}(x_1)\}$ and $\Phi^-_{\pi,r}(\boldsymbol{x}) = \{ux_0 \mid u \in \Phi^-_{\pi,r}(x_0)\}$.*

Proof. Let $\boldsymbol{x} = x_0x_1$ and $\boldsymbol{y} = y_0y_1$ be two different vertices of $L(G)$. If $d_{L(G)}(\boldsymbol{x}, \boldsymbol{y}) \leqslant \ell_{\pi,r}(G) + 1$, then $d_G(x_1, y_0) \leqslant \ell_{\pi,r}(G)$ and there is only one shortest path from x_1 to y_0 in G. Therefore, in $L(G)$, the shortest path from $\boldsymbol{x}$ to $\boldsymbol{y}$ is unique. Let us consider $\Phi^+_{\pi,r}(x_1) = \{w_1, \dots, w_s\}$ and $\Phi^-_{\pi,r}(y_0) = \{u_1, \dots, u_t\}$, where $1 \leqslant s,\ t \leqslant \pi$. If $d_{L(G)}(\boldsymbol{x}, \boldsymbol{y}) < \ell_{\pi,r}(G) + 1$, then $d_G(x_1, y_0) < \ell_{\pi,r}(G)$ and any non-shortest path from x_1 to y_0 with length at most $d(x_1, y_0) + r$ has its first vertex in $\Phi^+_{\pi,r}(x_1)$ and its last one in $\Phi^-_{\pi,r}(y_0)$. Therefore, all non-shortest paths from $\boldsymbol{x}$ to $\boldsymbol{y}$ with length at most $d(\boldsymbol{x}, \boldsymbol{y}) + r$ have their first vertices in $\Phi^+_{\pi,r}(\boldsymbol{x}) = \{x_1w_1, \dots, x_1w_s\}$ and their last ones in $\Phi^-_{\pi,r}(\boldsymbol{y}) = \{u_1y_0, \dots, u_ty_0\}$. If $\boldsymbol{x} = \boldsymbol{y} = x_0x_1$, and there is a cycle $\boldsymbol{x}\boldsymbol{x}_1 \dots \boldsymbol{x}_{h-1}\boldsymbol{x}$ with length $h \leqslant r$, then, there is a cycle $C = x_0x_1x_2 \dots x_{h-1}x_0$ in the digraph G. Since $\ell_{\pi,r}(G) \geqslant 1$, we have $x_2 \in \Phi^+_{\pi,r}(x_1)$ and $x_{h-1} \in \Phi^-_{\pi,r}(x_0)$. Then $\boldsymbol{x}_1 = x_1x_2$ is in $\Phi^+_{\pi,r}(\boldsymbol{x})$, and $\boldsymbol{x}_{h-1} = x_{h-1}x_0$ is in $\Phi^-_{\pi,r}(\boldsymbol{x})$. Therefore, $\ell_{\pi,r}(L(G)) \geqslant \ell_{\pi,r}(G) + 1$.

On the other hand, from the definition of $\ell_{\pi,r}(G)$, there exist vertices x, y of G such that

(1) $d(x, y) = \ell_{\pi,r}(G)$ and there exists a non-shortest path from x to y with length at most $d(x, y) + r$ such that its first vertex is not in $\Phi^+_{\pi,r}(x)$ or its last one is not in $\Phi^-_{\pi,r}(y)$; or
(2) $d(x, y) = \ell_{\pi,r}(G) + 1$ and there exist two different shortest paths from x to y.

Since $\delta \geqslant 2$, there exist vertices x_0, y_1 of G such that $\boldsymbol{x} = x_0 x$ and $\boldsymbol{y} = y y_1$ satisfy $d(\boldsymbol{x}, \boldsymbol{y}) = d(x, y) + 1$. Then,

(1) $d(\boldsymbol{x}, \boldsymbol{y}) = \ell_{\pi,r}(G) + 1$ and there exists a non-shortest path from $\boldsymbol{x}$ to $\boldsymbol{y}$ with length at most $d(\boldsymbol{x}, \boldsymbol{y}) + r$ such that its first vertex is not in $\Phi^+_{\pi,r}(\boldsymbol{x})$ or its last one is not in $\Phi^-_{\pi,r}(\boldsymbol{y})$; or
(2) $d(\boldsymbol{x}, \boldsymbol{y}) = \ell_{\pi,r}(G) + 2$ and there exist two different shortest paths from $\boldsymbol{x}$ to $\boldsymbol{y}$.

Therefore, $\ell_{\pi,r}(L(G)) \leqslant \ell_{\pi,r}(G) + 1$. □

Proposition 3.5. *Let G be a digraph with minimum degree $\delta \geqslant 2$. Let π and $r \geqslant 1$ be integers such that $0 \leqslant \pi \leqslant \delta - 2$ and $\ell_{\pi,r}(G) \geqslant 1$. Then, for any integer $k \geqslant 1$,*

(1) $\ell_{\pi,r}(L^k(G)) = \ell_{\pi,r}(G) + k$.
(2) *For any vertex $\boldsymbol{x} = x_0 x_1 \ldots x_k$ of $L^k(G)$, $\Phi^+_{\pi,r}(\boldsymbol{x}) = \{x_1 \ldots x_k w \mid w \in \Phi^+_{\pi,r}(x_k)\}$ and $\Phi^-_{\pi,r}(\boldsymbol{x}) = \{u x_0 \ldots x_{k-1} \mid u \in \Phi^-_{\pi,r}(x_0)\}$.*

Proof. This result is a direct consequence of Proposition 3.4. □

Definition 3.6. Let G be a digraph with minimum degree $\delta \geqslant 2$. Let π be an integer with $0 \leqslant \pi \leqslant \delta - 2$ and r a positive integer such that $\ell_{\pi,r}(G) \geqslant 1$. A (π, r)-*double detour* is a set of four paths $\{C_1, C_1', C_2, C_2'\}$ such that

- C_1 and C_1' are paths from x to f, with lengths s and s', respectively, where $s' \geqslant s$ and $s' \geqslant 1$. C_2 and C_2' are paths from f to y, with lengths t and t', respectively, where $t' \geqslant t$ and $t' \geqslant 1$. Besides, $\max\{s, t\} \geqslant 1$.
- If (x, x_1') is the first arc of C_1', then $x_1' \notin \Phi^+_{\pi,r}(x)$. If $s \neq 0$ and (x, x_1) is the first arc of C_1, then $x_1' \neq x_1$.
- If (y_1', y) is the last arc of C_2', then $y_1' \notin \Phi^-_{\pi,r}(y)$. If $t \neq 0$ and (y_1, y) is the last arc of C_2, then $x_1' \neq x_1$.

The *length* of a (π, r)-double detour is defined to be $s' + t'$. We define $M_{\pi,r}(G)$ as the minimum length of a (π, r)-double detour in G.

Proposition 3.7. *For any digraph G, $M_{1,1}(G) \geqslant 4$, and $M_{0,1}(G) \geqslant 4$ if G is loopless.*

Proof. Observe that $\ell_{1,1}(G) \geqslant 1$ for any digraph G. In effect, it is enough to consider $\Phi^+_{1,1}(x) = \Phi^-_{1,1}(x) = \{x\}$ for any vertex x such that the loop (x, x) is an arc of G. Let $\{C_1, C_1', C_2, C_2'\}$ be a $(1, 1)$-double detour in G. The paths C_1' and C_2' have length at least 2, even if C_1 or C_2 have length 0. Therefore, the length of any $(1, 1)$-double detour is at least 4. Equally, $\ell_{0,1}(G) \geqslant 1$ for any loopless digraph G. Obviously, $\Phi^+_{0,1}(x) = \Phi^-_{0,1}(x) = \emptyset$. It is not difficult to check that the length of any $(0, 1)$-double detour is at least 4 in any loopless digraph G. □

Proposition 3.8. *Let G be a digraph with minimum degree $\delta \geqslant 2$. Let π be an integer with $0 \leqslant \pi \leqslant \delta - 2$ and r a positive integer such that $\ell_{\pi,r}(G) \geqslant 1$. Then, for any positive integer k, $M_{\pi,r}(L^k(G)) = M_{\pi,r}(G) + k$.*

Proof. Since $L^k(G)=L(L^{k-1}(G))$, it is enough to prove the proposition for $k=1$. Let $\{C_1,C_1',C_2,C_2'\}$ be a (π,r)-double detour in $L(G)$ with length $s'+t'$, where C_1 and C_1' are paths from $\boldsymbol{x}=x_0x_1$ to $\boldsymbol{f}=f_0f_1$ and C_2 and C_2' are paths from $\boldsymbol{f}=f_0f_1$ to $\boldsymbol{y}=y_0y_1$. We may assume that $s\geqslant 1$. Let us consider in G the paths $\{\mathscr{C}_1,\mathscr{C}_1',\mathscr{C}_2,\mathscr{C}_2'\}$, where $\mathscr{C}_1$ and $\mathscr{C}_1'$ are the paths from x_1 to f_0 that are obtained, respectively, from C_1 and C_1'. The first arcs of $\mathscr{C}_1$ and $\mathscr{C}_1'$ are different and, by Proposition 3.4, the second vertex of $\mathscr{C}_1'$ is not in $\Phi^+_{\pi,r}(x_1)$. Equally, one obtains from C_2 and C_2' the paths $\mathscr{C}_2$ and $\mathscr{C}_2'$ from f_0 to y_0 (if $t=0$, then $f_0=y_0$ and $\mathscr{C}_2$ is also a path with length 0). As before, the last but one vertices of $\mathscr{C}_2$ and $\mathscr{C}_2'$ are different and the last but one vertex of $\mathscr{C}_2'$ is not in $\Phi^-_{\pi,r}(y_0)$. Therefore, we have found a (π,r)-double detour in G with length $s'+t'-1$.

Therefore, if there is a (π,r)-double detour in $L(G)$ with length $s'+t'$, then there exists a (π,r)-double detour in G with length $s'+t'-1$. On the other hand, If there exists a (π,r)-double detour in G with length h, formed by paths between the vertices x_1 and y_0, the corresponding paths from x_0x_1 to y_0y_1 form a (π,r)-double detour in $L(G)$ with length $h+1$. □

Lemma 3.9. *Let G be a digraph with minimum degree $\delta\geqslant 2$. Let π and $r\geqslant 1$ be two integers with $0\leqslant\pi\leqslant\delta-2$ and $\ell_{\pi,r}(G)\geqslant 1$. Let x,y,f be any three vertices. If $x_1\in\Gamma^+(x)-\Phi^+_{\pi,r}(x)$, $x_1\neq v(x\to f)$ and $y_1\in\Gamma^-(y)-\Phi^-_{\pi,r}(y)$, $y_1\neq v(y\leftarrow f)$, then $d(x_1,f)+d(f,y_1)\geqslant M_{\pi,r}(G)-2$.*

Proof. Since $x_1\notin\Phi^+_{\pi,r}(x)$, $x_1\neq v(x\to f)$ and $y_1\notin\Phi^-_{\pi,r}(y)$, $y_1\neq v(y\leftarrow f)$, we can consider a (π,r)-double detour in G with C_1 a shortest path from x to f, C_2 a shortest path from f to y, $C_1'=xx_1\ldots f$ and $C_2'=f\ldots y_1y$. Then, $M_{\pi,r}(G)\leqslant d(x_1,f)+d(f,y_1)+2$. □

The following result can be proved analogously.

Lemma 3.10. *Let G be a digraph with minimum degree $\delta\geqslant 2$. Let π and $r\geqslant 1$ be two integers with $0\leqslant\pi\leqslant\delta-2$ and $\ell_{\pi,r}(G)\geqslant 1$. Let x,y be two vertices and (f,g) an arc. Let us consider $x_1\in\Gamma^+(x)-\Phi^+_{\pi,r}(x)$ such that $x_1\neq v(x\to f)$ if $x\neq f$ and $x_1\neq g$ if $x=f$, and $y_1\in\Gamma^-(y)-\Phi^-_{\pi,r}(y)$ such that $y_1\neq v(y\leftarrow g)$ if $y\neq g$ and $y_1\neq f$ if $y=g$. Then, $d(x_1,f)+d(g,y_1)\geqslant M_{\pi,r}-3$.*

4. The bounds

In this section we present upper bounds for both, vertex and arc-diameter vulnerability of iterated line digraphs, making use of the results of Section 3.

Theorem 4.1. *Let G be a digraph with minimum degree $\delta\geqslant 2$ and diameter $D=D(G)$. Let π, r be a pair of integers such that $0\leqslant\pi\leqslant\delta-2$, $r\geqslant 1$ and $\ell_{\pi,r}=\ell_{\pi,r}(G)\geqslant 1$. Let* *us consider $M_{\pi,r} = M_{\pi,r}(G)$. Then, for any integer k such that $k \geqslant D - 2\ell_{\pi,r} + 1$, the s-vertex-diameter-vulnerability of $L^k(G)$ verifies*

$$K(s, L^k(G)) \leqslant D(L^k(G)) + C$$

for any $s = 1, \ldots, \delta - \pi - 1$, where

$$C = \max\left\{ \left\lceil \frac{D - M_{\pi,r} + 3 + 2r}{r} \right\rceil, 2\left\lceil \frac{D - \ell_{\pi,r}}{r} \right\rceil \right\}.$$

Proof. Let F be a non-empty set of faulty vertices of $L^k(G)$, $|F| = s \leqslant \delta - \pi - 1$. Let x, y be two different vertices of $L^k(G)$ which are not in F. As $|F| \leqslant \delta - \pi - 1$, there exist $x_1 \in \Gamma^+(x) - \Phi^+_{\pi,r}(x) - v(x \to F)$ and $y_1 \in \Gamma^-(y) - \Phi^-_{\pi,r}(y) - v(y \leftarrow F)$. By Proposition 3.8 and Lemma 3.9,

$$d(x_1, f) + d(f, y_1) \geqslant M_{\pi,r} + k - 2$$

for all $f \in F$. By Lemma 3.2 and Proposition 3.5, for any $f \in F$,

$$d(x_1, f) \geqslant \min\{d(x, f) + r, \ell_{\pi,r} + k\}.$$

Equally,

$$d(f, y_1) \geqslant \min\{d(f, y) + r, \ell_{\pi,r} + k\}$$

for all $f \in F$. By Lemma 3.3, for any pair of integers m, n, there exist paths $x_1 x_2 \ldots x_m$ and $y_n \ldots y_2 y_1$, with $x_i,\ y_i \notin F$, such that, for any $f \in F$,

$$d(x_m, f) \geqslant \min\{d(x_1, f) + r(m-1), \ell_{\pi,r} + k\}$$

and

$$d(f, y_n) \geqslant \min\{d(f, y_1) + r(n-1), \ell_{\pi,r} + k\}$$

Let us consider m and n such that $mr,\ nr \geqslant D - \ell_{\pi,r}$ and

$$m + n = \max\left\{ \left\lceil \frac{D - M_{\pi,r} + 3 + 2r}{r} \right\rceil, 2\left\lceil \frac{D - \ell_{\pi,r}}{r} \right\rceil \right\}.$$

By combining the inequalities above and taking into account that $k \geqslant D - 2\ell_{\pi,r} + 1$, it is not difficult to check that $d(x_m, f) + d(f, y_n) \geqslant D + k + 1 = D(L^k(G)) + 1$ for any $f \in F$. Then, a shortest path from x_m to y_n does not contain any vertex in F. Therefore, we have found a path from x to y with length at most $D + k + m + n = D + k + C$ avoiding F. □

Bounds on the arc-diameter-vulnerability are found in a similar way.

Theorem 4.2. *Let G be a digraph with minimum degree $\delta \geqslant 2$ and diameter $D = D(G)$. Let π, r be a pair of integers such that $0 \leqslant \pi \leqslant \delta - 2$, $r \geqslant 1$ and $\ell_{\pi,r} = \ell_{\pi,r}(G) \geqslant 1$. Let us consider $M_{\pi,r} = M_{\pi,r}(G)$. Then, for any integer k such that $k \geqslant D - 2\ell_{\pi,r}$, the s-arc-diameter-vulnerability of $L^k(G)$ verifies*

$$\Lambda(s, L^k(G)) \leqslant D(L^k(G)) + C$$

for any $s = 1, \dots, \delta - \pi - 1$, *where*

$$C = \max\left\{\left\lceil \frac{D - M_{\pi,r} + 3 + 2r}{r} \right\rceil, 2\left\lceil \frac{D - \ell_{\pi,r}}{r} \right\rceil\right\}.$$

Proof. Let $\mathscr{F}$ be a non-empty set of faulty arcs of $L^k(G)$, $\mathscr{F} = \{e_1, \dots, e_s\}$, where $1 \leqslant s \leqslant \delta - \pi - 1$ and $e_i = (f_i, g_i)$ for $i = 1, \dots, s$. Let us consider the sets of vertices $F = \{f_1, \dots, f_s\}$ and $G = \{g_1, \dots, g_s\}$. Let x, y be two different vertices of $L^k(G)$ and consider $F_1 = F - \{x\}$ and $G_1 = G - \{y\}$. Since $|\mathscr{F}| \leqslant \delta - \pi - 1$, there exists a vertex $x_1 \in \Gamma^+(x) - \Phi^+_{\pi,r}(x)$ such that $x_1 \neq v(x \to F_1)$ and $x_1 \neq g_i$ if $x = f_i$. Equally, there exists a vertex $y_1 \in \Gamma^-(y) - \Phi^-_{\pi,r}(y)$ such that $y_1 \neq v(y \leftarrow G_1)$ and $y_1 \neq f_i$ if $y = g_i$. Observe that $x_1 \notin F$ and $y_1 \notin G$. By Proposition 3.8 and Lemma 3.10,

$$d(x_1, f) + d(g, y_1) \geqslant M_{\pi,r} + k - 3$$

for any arc $(f, g) \in \mathscr{F}$. Applying now Lemma 3.2 and Proposition 3.5, we have

$$d(x_1, f) \geqslant \min\{d(x, f) + r, \ell_{\pi,r} + k\}$$

for any $f \in F$ (it is possible that $d(x, f) = 0$). Equally,

$$d(g, y_1) \geqslant \min\{d(g, y) + r, \ell_{\pi,r} + k\}$$

for any $g \in G$. From Lemma 3.3, for any pair of integers m, n, there exist paths $x_1 x_2 \dots x_m$ and $y_n \dots y_2 y_1$ such that

$$d(x_m, f) \geqslant \min\{d(x_1, f) + r(m - 1), \ell_{\pi,r} + k\}$$

for any $f \in F$ and

$$d(g, y_n) \geqslant \min\{d(g, y_1) + r(n - 1), \ell_{\pi,r} + k\}$$

for any $g \in G$. Let us consider the same values of m and n as in the proof of Theorem 4.1. It is not difficult to check that $d(x_m, f) + d(g, y_n) \geqslant D + k = D(L^k(G))$ for any arc $(f, g) \in \mathscr{F}$. Then, a shortest path from x_m to y_m cannot contain any arc in $\mathscr{F}$. Therefore, there exists a path $x x_1 \dots x_m \dots y_n \dots y_1 y$ from x to y with length at most $D(L^k(G)) + m + n$ that avoids the set of faulty arcs $\mathscr{F}$. $\square$

Theorems 4.1 and 4.2 in [10] are a consequence of Proposition 3.7 and the following corollary, which is proved by taking $\pi = 1$ and $r = 1$ in the previous theorems

Corollary 4.3. *Let G be a digraph with minimum degree $\delta > 2$, diameter $D = D(G)$ and $\ell_{1,1} = \ell_{1,1}(G) = \ell^*_1(G)$. Then,*

- $K(s, L^k(G)) \leqslant D(L^k(G)) + C$, *if* $k \geqslant D - 2\ell_{1,1} + 1$,
- $\Lambda(s, L^k(G)) \leqslant D(L^k(G)) + C$, *if* $k \geqslant D - 2\ell_{1,1}$

for $s = 1, \dots, \delta - 2$, *where* $C = \max\{D - M_{1,1} + 5, 2(D - \ell_{1,1})\}$.

If we take $\pi = 0$ and $r = 1$, we obtain the following result, from which Theorems 3.1 and 3.2 in [10] follow.

Corollary 4.4. *Let G be a digraph without loops and with minimum degree $\delta \geqslant 2$, diameter $D = D(G)$ and $\ell_{0,1} = \ell_{0,1}(G) = \ell_0(G)$. Then,*

- $K(s, L^k(G)) \leqslant D(L^k(G)) + C$, *if* $k \geqslant D - 2\ell_{0,1} + 1$,
- $\Lambda(s, L^k(G)) \leqslant D(L^k(G)) + C$, *if* $k \geqslant D - 2\ell_{0,1}$

for $s = 1, \ldots, \delta - 1$, *where* $C = \max\{D - M_{0,1} + 5, 2(D - \ell_{0,1})\}$.

5. Applications

A generalized p-cycle is a digraph whose set of vertices is partitioned in p parts that are cyclically ordered in such a way that the vertices in one part are only adjacent to vertices in the next cycle. The Kautz generalized cycles KGC $(p, d, d^{p+k} + d^k)$ were proposed in [6] as a solution to the (d, D)-problem restricted to generalized cycles. It was proved there, that if $2p - 1 \leqslant D \leqslant 3p - 2$, the order of the digraph KGC $(p, d, d^{p+k} + d^k)$ is the largest that a p-cycle could have with degree d and diameter D. This family is also a generalization of other models for interconnection networks. For example, $K(d, D)$, the Kautz digraph of degree d and diameter D is the same as KGC $(1, d, d^D + d^{D-1})$. Also the generalized Kautz or Imase–Itoh digraph [7,8], $GK(d, n)$, can be defined as KGC $(1, d, n)$. The bipartite digraphs BD(d, n) introduced in [4] for the (d, D)-problem over bipartite digraphs, coincides with KGC $(2, d, d^{D-p+1} + d^{D-2p+1})$.

The diameter-vulnerability of some particular families of Kautz generalized cycles has been calculated by finding disjoint paths between any pair of vertices. For example, for the de Bruijn and Kautz digraphs [9], the bipartite digraphs BD(d, n) [11] and in general, for the digraphs KGC $(p, d, d^{p+k} + d^k)$ [3].

The Kautz generalized cycles are iterated line digraphs. More exactly, KGC $(p, d, d^{p+k} + d^k) = L^k(\text{KGC } (p, d, d^p + 1))$. The digraph KGC $(p, d, d^p + 1)$ has diameter $D = 2p - 1$ and parameters $M_{0,p-1} = 2p + 2$, $\ell_{0,p-1} = p$ (we refer the reader to [6,3] for details). With these values, the bound in [10] is

$$K(s, \text{KGC } (p, d, d^{p+k} + d^k)), \Lambda(s, \text{KGC } (p, d, d^{p+k} + d^k)) \leqslant D(L^k(G)) + 4(p - 1)$$

for $s = 1, \ldots, d - 1$. Instead, by the results obtained in Section 4:

$$K(s, \text{KGC } (p, d, d^{p+k} + d^k)), \Lambda(s, \text{KGC } (p, d, d^{p+k} + d^k)) \leqslant D(L^k(G)) + 2$$

for $s = 1, \ldots, d - 1$. These bounds coincide with the exact values given in [11,3] for the diameter vulnerability of BD(d, n) and KGC $(p, d, d^{p+k} + d^k)$, respectively.

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