Power domination and zero forcing

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Abstract

In this paper, we use the relationship between zero forcing and power domination established in Dean et al. to obtain the inequality $\left\lceil \frac{\mathrm{Z}(G)}{\Delta(G)} \right\rceil \le \gamma_P(G)$ where $\gamma_P(G)$ is the the power domination number of G, $\mathrm{Z}(G)$ is the zero forcing number of G, and $\Delta(G)$ is the maximum degree of G. We apply this to establish new results for both parameters, including the power domination number for the Cartesian product of two cycles and the zero forcing number of the lexicographic product of regular graphs. We also establish bounds on the effect of a graph operation (vertex and edge deletion, edge contraction, and edge subdivision) on the power domination number.

Keywords power domination, zero forcing, maximum nullity, minimum rank
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1 Introduction

Electric power companies need to monitor the state of their networks continuously. One method of monitoring a network is to place Phase Measurement Units (PMUs) at selected locations in the system, called electrical nodes or buses, where transmission lines, loads, and generators are connected. A PMU placed at an electrical node measures the voltage at the node and all current phasors at the node [2]. The placement of PMUs at all nodes of a network is a trivial solution to the monitoring problem. Because of the cost of a PMU, the trivial solution is not feasible, and it is important to minimize the number of PMUs used while maintaining the ability to observe the entire system.

This problem was first studied in terms of graphs by Haynes et al. in [12]. Indeed, an electric power network can be modeled by a graph where the vertices represent the electric nodes and the edges are associated with the transmission lines joining two electrical nodes. In this model, the power domination problem in graphs consists of finding a minimum set of vertices from where the entire graph can be observed according to certain rules. In terms of the physical network, such a minimum set of vertices will provide the locations where the PMUs should be placed in order to monitor the entire graph at minimum cost. A PMU measures the voltage and phase angle at the vertex where it is located and also at other vertices or edges according to certain propagation rules (see Section 1.1 for the formal definitions of this and other terms). Since its introduction in [12], the power domination number and variations have generated considerable interest (see, for example, [4, 5, 8, 9]).

As was pointed out in [7], a careful examination of the power domination definition leads naturally to the study of zero forcing. The zero forcing number was introduced in [1] as an upper bound for the

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maximum nullity of real symmetric matrices whose nonzero pattern of off-diagonal entries is described by a given graph, and independently by mathematical physicists studying control of quantum systems.

In Section 2 we use the connection between power domination and zero forcing that was established in [7] to obtain a lower bound for the power domination number γ_P as a function of the zero forcing number Z, or equivalently the upper bound on the zero forcing number as a function of the power domination number (Theorem 2.2 below). We then use this relationship to prove new results for both parameters. We show that $\gamma_P(C_n \,\square\, C_m) = \left\lceil \frac{n}{2} \right\rceil$ for $m \geq n \geq 3$, where C_n is a cycle of order n. Next we obtain a bound on the zero forcing number $Z(G * H)$, where G and H are regular graphs and $G * H$ is the lexicographic product of G and H, and use this result to compute $\mathrm{Z}(K_n * C_m) = (n-1)m + 2$, for $n \geq 2, m \geq 3$, where K_n is a complete graph of order n.

In Section 3, we discuss the effect of various graph operations on the power domination number. First, we prove that deleting a vertex from a graph can reduce the power domination number by at most one (or increase it by any amount). Next, we prove that the power domination number of a graph obtained by deleting an edge or contracting an edge is either one less, the same, or one greater than the power domination number of the original graph. Finally, we prove that the power domination number of a graph obtained by subdividing an edge is either the power domination number of the original graph or one greater than that number.

1.1 Power domination and zero forcing definitions

A *graph* $G = (V, E)$ is an ordered pair formed by a finite nonempty set of *vertices* $V = V(G)$ and a set of *edges* $E = E(G)$ containing unordered pairs of distinct vertices (that is, all graphs are simple and undirected). The *order* of G is denoted by $|G| := |V(G)|$. For any vertex $v \in V$, the neighborhood of v is the set $N(v) = \{u \in V : \{u, v\} \in E\}$ (or $N_G(v)$ if G is not clear from context), and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. Similarly, for any set of vertices S, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$.

A vertex v in a graph G is said to *dominate* itself and all vertices adjacent to v in G. A set of vertices S is a *dominating set* of G if every vertex of G is dominated by a vertex in S. The cardinality of a dominating set of minimum cardinality is the *domination number* of G and is denoted by $\gamma(G)$.

In [12] the authors introduced the related concept of power domination by presenting propagation rules in terms of vertices and edges in a graph. In this paper we will use a simplified version of the propagation rules that is equivalent to the original [8]. For a set S of vertices in a graph G, define $PD(S) \subseteq V(G)$ recursively:

1. $PD(S) := N[S] = S \cup N(S)$.
2. While there exists $v \in PD(S)$ such that $|N(v) \cap (V(G) \setminus PD(S))| = 1$:
   $PD(S) := PD(S) \cup N(v)$.

We say that a set $S \subseteq V(G)$ is a *power dominating set* of a graph G if at the end of the process above $PD(S) = V(G)$. A *minimum power dominating set* is a power dominating set of minimum cardinality, and the *power domination number* $\gamma_P(G)$ of G is the cardinality of a minimum power dominating set.

The *color change rule* is: If u is a blue vertex and exactly one neighbor w of u is white, then change the color of w to blue. We say u forces w and denote this by $u \to w$. A *zero forcing set* for G is a subset of vertices B such that when the vertices in B are colored blue and the remaining vertices are colored white initially, repeated application of the color change rule can color all vertices of G blue. A *minimum zero forcing set* is a zero forcing set of minimum cardinality, and the *zero forcing number* $\mathrm{Z}(G)$ of G is the cardinality of a minimum zero forcing set. The next observation is central to this paper.

Observation 1.1. [7] *The power domination process on a graph G can be described as choosing a set $S \subseteq V(G)$ and applying the zero forcing process to the closed neighborhood $N[S]$ of S. The set S is a power dominating set of G if and only if $N[S]$ is a zero forcing set for G.*

The *degree* of a vertex v, denoted by $\deg v$, is the cardinality of the set $N(v)$. The *maximum* and *minimum degree* of G are defined as $\Delta(G) = \max\{\deg v : v \in V\}$ and $\delta(G) = \min\{\deg v : v \in V\}$, respectively. A graph G is *regular* if $\delta(G) = \Delta(G)$. The next observation is well known (and immediate since zero forcing on G cannot start without at least $\delta(G)$ blue vertices).

Observation 1.2. *For every graph G, $\delta(G) \le \mathrm{Z}(G)$.*

The set of all $n \times n$ real symmetric matrices is denoted by $S_n(\mathbb{R})$. For $A = [a_{ij}] \in S_n(\mathbb{R})$, the *graph* of A, denoted by $\mathcal{G}(A)$, is the graph with vertices $\{1, \dots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0,\ 1 \le i < j \le n\}$. Note that the diagonal of A is ignored in determining $\mathcal{G}(A)$. The set of symmetric matrices described by an arbitrary graph G of order n is defined as $\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}$. The *maximum nullity* of G is $\mathrm{M}(G) = \max\{\mathrm{null}\, A : A \in \mathcal{S}(G)\}$, and the *minimum rank* of G is $\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : A \in S(G)\}$; clearly $\mathrm{M}(G) + \mathrm{mr}(G) = |G|$. The term 'zero forcing' comes from using the forcing process to force zeros in a null vector of a matrix $A \in \mathcal{S}(G)$, implying $\mathrm{M}(G) \le \mathrm{Z}(G)$ [1].

1.2 Graph definitions and notation

Let n, p, q be positive integers. A *path* of order n is a graph P_n with vertices $V(P_n) = \{x_i : 0 \le i \le n-1\}$ and edges $E(P_n) = \{\{x_i, x_{i+1}\} : 0 \le i \le n-2\}$. If $n \ge 3$, the *cycle* of order n is the graph C_n with vertex set $V(C_n) = \{x_i : 0 \le i \le n-1\}$ and edge set $E(C_n) = \left\{\{x_i, x_{(i+1) \bmod n}\} : 0 \le i \le n-1\right\}$. A *complete graph* of order n is a graph K_n with $V(K_n) = \{x_i : 0 \le i \le n-1\}$ and $E(K_n) = \{\{x_i, x_j\} : 0 \le i < j \le n-1\}$. A *complete bipartite graph* with partite sets X and Y of orders p and q is a graph $K_{p,q}$ with $V(K_{p,q}) = X \cup Y$ where $X = \{x_i : 0 \le i \le p-1\}$ and $Y = \{y_i : 0 \le i \le q-1\}$ are disjoint, and $E(K_{p,q}) = \{\{x_i, y_j\} : 0 \le i \le p-1, 0 \le j \le q-1\}$.

For a graph $G = (V, E)$ and $W \subseteq V$, the *induced subgraph* $G[W]$ is the graph with vertex set W and edge set $\{\{w, u\} \in E : w, u \in W\}$. The graph induced by $W = V \setminus \{v\}$ is also denoted by $G - v$. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be disjoint graphs. The *corona* of G and H, denoted by $G \circ H$, is the disjoint union of G with $|G|$ copies of H such that the i^{th} vertex v_i of G is joined to all the vertices in the i^{th} copy of H. The *Cartesian product* of G and H, denoted by $G \,\square\, H$, has the vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in $G \,\square\, H$ if either (1) $g = g'$ and $\{h, h'\} \in E(H)$, or (2) $h = h'$ and $\{g, g'\} \in E(G)$. The *lexicographic product* of G and H, denoted by $G * H$, has the vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in $G * H$ if and only if $\{g, g'\} \in E(G)$, or $g = g'$ and $\{h, h'\} \in E(H)$.

For a graph G with no edges, $\mathrm{Z}(G) = \gamma_P(G) = \gamma(G) = |G|$, so we focus our attention on graphs with edges.

2 Zero forcing lower bound for power domination number

The power domination number of several families of graphs has been determined using a two-step process: finding an upper bound and a lower bound. The upper bound is usually obtained by providing a pattern to construct a set, together with a proof that constructed set is a power dominating set. The lower bound is usually found by exploiting structural properties of the particular family of graphs, and it usually consists of a very technical and lengthy process. Therefore, finding good general lower bounds for the power domination number is an important problem.

An effort in that direction is the work by Stephen et al. [15, Theorem 3.1] in which a general lower bound is presented and successfully applied to finding the power domination number of some graphs modeling chemical structures. However, their bound depends heavily on the choice of a family of subgraphs satisfying certain properties. While in some graphs it is possible to find families of subgraphs that yield good lower bounds, in other graphs it is not. In fact, in some graphs the only family of subgraphs satisfying the conditions of [15, Theorem 3.1] is the family that consists of the graph itself, which yields a trivial lower bound, as is the case for paths P_n, $n \ge 2$.

A result of Dean et al. [7], stated in Theorem 2.1 below, provides an immediate lower bound for the power domination number of a regular graph when the zero forcing number is known. This was the approach used on hypercubes in [7]. We use this theorem to establish a lower bound on power dominating number in terms of zero forcing number and maximum degree for any graph with an edge in Theorem 2.2 below.

Theorem 2.1. [7, Lemma 2] *Let G be a graph with no isolated vertices and let $S = \{u_1, \dots, u_t\}$ be a power dominating set for G. Then $\mathrm{Z}(G) \le \sum_{i=1}^{t} \deg u_i$.*

The next theorem, which follows from Theorem 2.1, can be used to map zero forcing results to power dominating results and vice versa (as is done in Sections 2.1 and 2.2 below).

Theorem 2.2. *Let G be a graph that has an edge. Then* $\left\lceil \frac{\mathrm{Z}(G)}{\Delta(G)} \right\rceil \le \gamma_P(G)$ *and this bound is tight.*

Proof. Suppose G has connected components $G_1, \dots, G_h$. First suppose G_i is a component that has an edge, so G_i does not have isolated vertices. Choose a minimum power dominating set $S_i = \{u_1^{(i)}, \dots, u_{t_i}^{(i)}\}$ for G_i, so $t_i = \gamma_P(G_i)$. Then by Theorem 2.1, $\mathrm{Z}(G_i) \le \sum_{i=1}^{t_i} \deg u_i^{(i)} \le t_i \Delta(G_i) = \gamma_P(G_i)\Delta(G_i)$. Thus $\gamma_P(G_i) \ge \left\lceil \frac{\mathrm{Z}(G_i)}{\Delta(G_i)} \right\rceil \ge \left\lceil \frac{\mathrm{Z}(G_i)}{\Delta(G)} \right\rceil$.

Since G has an edge, $\Delta(G) \ge 1$, and so $\gamma_P(G_j) \ge \left\lceil \frac{\mathrm{Z}(G_j)}{\Delta(G)} \right\rceil$ for every component G_j of G (including isolated vertices). Thus

$$\gamma_P(G) = \sum_{i=1}^{h} \gamma_P(G_i) \ge \sum_{i=1}^{h} \left\lceil \frac{\mathrm{Z}(G_i)}{\Delta(G)} \right\rceil \ge \left\lceil \sum_{i=1}^{h} \frac{\mathrm{Z}(G_i)}{\Delta(G)} \right\rceil = \left\lceil \frac{\sum_{i=1}^{h} \mathrm{Z}(G_i)}{\Delta(G)} \right\rceil = \left\lceil \frac{\mathrm{Z}(G)}{\Delta(G)} \right\rceil$$

Since $\mathrm{Z}(K_n) = \Delta(K_n) = n-1$ and $\gamma_P(K_n) = 1$, the bound is tight. □

The next corollary is immediate from the fact that $\mathrm{M}(G) \le \mathrm{Z}(G)$ [1]. Although weaker than Theorem 2.2, Corollary 2.3 can sometimes be applied using a well known matrix such as the adjacency or Laplacian matrix of the graph, even if $\mathrm{M}(G)$ and $\mathrm{Z}(G)$ are not known.

Corollary 2.3. *For a graph G that has an edge and any matrix $A \in \mathcal{S}(G)$,* $\left\lceil \frac{\mathrm{null}\, A}{\Delta(G)} \right\rceil \le \gamma_P(G)$.

2.1 Application to computation of power domination number of families of graphs

In this section we apply Theorem 2.2 and results about the zero forcing number to obtain results about the power domination number To establish the value of $\gamma_P(C_n \,\Box\, C_m)$, we need to establish the zero forcing number of Cartesian products of cycles.

Theorem 2.4. *For $m \ge n \ge 3$,*

$$\mathrm{M}(C_n \,\Box\, C_m) = \mathrm{Z}(C_n \,\Box\, C_m) = \begin{cases} 2n-1 & \text{if } m = n \text{ and } n \text{ is odd} \\ 2n & \text{otherwise.} \end{cases}$$

Proof. For $m = n \ge 3$, by [6, Theorem 2.18] $\mathrm{M}(C_n \,\Box\, C_n) = \mathrm{Z}(C_n \,\Box\, C_n) = n + 2\left\lfloor \frac{n}{2} \right\rfloor$, so $\mathrm{M}(C_n \,\Box\, C_m) = \mathrm{Z}(C_n \,\Box\, C_n) = 2n-1$ for n odd and $\mathrm{M}(C_n \,\Box\, C_n) = \mathrm{Z}(C_n \,\Box\, C_n) = 2n$ for n even.

So assume $m > n \ge 3$. It is easy to see that the vertices of two consecutive cycles form a zero forcing set, so $\mathrm{Z}(C_n \,\Box\, C_m) \le 2n$. To complete the proof we construct a matrix in $\mathcal{S}(C_n \,\Box\, C_m)$ with nullity $2n$, so $2n \le \mathrm{M}(C_n \,\Box\, C_m) \le \mathrm{Z}(C_n \,\Box\, C_m) \le 2n$.

A standard way to construct matrices of maximum nullity for a Cartesian product of graphs is to use the *Kronecker* or *tensor* product of matrices [1, Observation 3.5]: If A is an $n \times n$ real matrix and B is an $m \times m$ real matrix, then $A \otimes B$ is the $n \times n$ block matrix whose ijth block is the $m \times m$ matrix $a_{ij}B$. If G and H are graphs of orders n and m, respectively, with $A \in \mathcal{S}(G)$ and $B \in \mathcal{S}(H)$, then $A \otimes I_m - I_n \otimes B \in \mathcal{S}(G \,\Box\, H)$. If $\mathbf{x}$ is an eigenvector of A for eigenvalue λ and $\mathbf{y}$ is an eigenvector of B for eigenvalue ν, then $\mathbf{x} \otimes \mathbf{y}$ is an eigenvector $A \otimes I_m - I_n \otimes B$ for eigenvalue $\lambda - \nu$. Since a real symmetric matrix has a full set of eigenvectors, the multiplicity of $\lambda - \nu$ is $\mathrm{mult}_A(\lambda)\,\mathrm{mult}_B(\nu)$.

Let $k = \left\lceil \frac{n}{2} \right\rceil$. Let A be the matrix obtained from the adjacency matrix of C_n by changing one pair of symmetrically placed entries from 1 to -1. Then the distinct eigenvalues of A are $\mu_i = 2\cos\frac{\pi(2i-1)}{n}$, $i = 1, \dots, k$, each with multiplicity 2 except that $\mu_k = -2$ has multiplicity 1 when n is odd [1, Theorem 3.8]. Once it is established that there is a matrix $B \in \mathcal{S}(C_m)$ such that μ_i is an eigenvalue of B of multiplicity 2 for $i = 1, \dots, k$, then $A \otimes I_m - I_n \otimes B$ has eigenvalue zero with multiplicity $2n$, because every eigenvalue of A has a corresponding eigenvalue of B with multiplicity 2.

It remains to establish the existence of a matrix $B \in \mathcal{S}(C_m)$ such that μ_i is an eigenvalue of B of multiplicity 2 for $i = 1, \ldots, k$. In Ferguson [11, Theorem 4.3] it is shown that for any set of real numbers $\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 \geq \lambda_5 > \ldots$ there is a periodic Jacobi matrix $B \in \mathcal{S}(C_m)$. Thus for m odd or $m \geq n+2$ we can choose $\lambda_{2i} = \lambda_{2i+1} = \mu_i$. Hall [13] showed that for $m = 2k$ and any $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ there is a matrix $B \in \mathcal{S}(C_m)$ with $\text{mult}_B(\lambda_i) = 2$ for $i = 1, \ldots, k$. This concludes the proof. □

Theorem 2.5. *For $m \geq n \geq 3$,*

$$\gamma_P(C_n \,\square\, C_m) = \left\lceil \frac{n}{2} \right\rceil.$$

Proof. Since $\Delta(C_n \,\square\, C_m) = 4$ and $\lceil \frac{2n-1}{4} \rceil = \lceil \frac{n}{2} \rceil$ for n odd, it follows from Theorems 2.2 and 2.4 that

$$\left\lceil \frac{n}{2} \right\rceil \leq \gamma_P(C_n \,\square\, C_m) \tag{1}$$

for $m \geq n \geq 3$.

In [4, Theorem 4.2] it was proved that for $m \geq n$,

$$\gamma_P(C_n \,\square\, C_m) \leq \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 2 \mod 4 \\ \lceil \frac{n+1}{2} \rceil & \text{otherwise.} \end{cases}$$

By (1),

$$\gamma_P(C_n \,\square\, C_m) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 2 \mod 4 \\ \lceil \frac{n}{2} \rceil \text{ or } \lceil \frac{n+1}{2} \rceil & \text{otherwise.} \end{cases}$$

If n is odd, then $\lceil \frac{n}{2} \rceil = \lceil \frac{n+1}{2} \rceil$. Finally, suppose $n = 4k$ and denote the vertices of $C_n \,\square\, C_m$ by (a, b) with $a \in \mathbb{Z}_n$ and $b \in \mathbb{Z}_m$. It is well known that $B = \{(a, 0), (a, 1) : 0 \leq a \leq n-1\}$ is a zero forcing set for $C_n \,\square\, C_m$. Since $S = \{(4i, 0), (4i+2, 1) : i = 0, \ldots, k-1\}$ dominates B, S is a power dominating set of $\frac{n}{2}$ elements. Thus $\gamma_P(C_n \,\square\, C_m) = \lceil \frac{n}{2} \rceil$. □

Many of the proofs of the values of the power domination number for families can be simplified by application of the relationship between power domination and zero forcing. Here we give two brief examples of simplifying a proof, beginning with a corollary of Theorem 2.2.

Corollary 2.6. [4, Lemma 3.1] *For $n \geq 3$, $\gamma_P(C_n \circ K_1) \geq \lceil \frac{n}{6} \rceil$.*

Proof. The result follows from $\text{Z}(C_n \circ K_1) = \lceil \frac{n}{2} \rceil$ (see [1]) and $\Delta(C_n \circ K_1) = 3$. A straightforward verification shows $\left\lceil \frac{\lceil \frac{n}{2} \rceil}{3} \right\rceil = \lceil \frac{n}{6} \rceil$. □

A *spider* or *generalized star* is a tree formed from a $K_{1,n}$ by subdividing any number of its edges any number of times (the precise definition of edge subdivision is given in Section 3.3). The *spider number* $\text{sp}(G)$ is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a spider (such a partition is called a *spider cover*). For a given zero forcing set, apply the color change rule as needed to color all vertices blue, listing the forces in the order in which they were performed. This list is called a *chronological list of forces*, often denoted by $\mathcal{F}$. Consider a vertex v in a power dominating set S. Perform zero forcing using $N[S]$ and record the chronological list of forces $\mathcal{F}$. Clearly the subgraph consisting of v, the neighbors of v together with the edges connecting each of them to v, and the vertices and edges in the forcing paths associated with the neighbors of v defines a spider subgraph of G. This subgraph is called the *forcing spider* of v (with respect to S and $\mathcal{F}$) and is denoted by T_v. The *set of forcing spiders* of S for a chronological list $\mathcal{F}$ is $\{T_v : v \in S\}$. The next remark gives a simpler proof of [12, Lemma 10].

Remark 2.7. Suppose T is a tree, S is a power dominating set of T, and $\mathcal{F}$ is a chronological list of forces of $N[S]$. For $v \in S$, the forcing spider of v is an induced subgraph because any connected subgraph of a tree is an induced subgraph. Thus the set of forcing spiders of S for a chronological list $\mathcal{F}$ is a spider cover of G, so $\text{sp}(T) \leq \gamma_P(T)$. (Note that for an arbitrary graph the forcing spider of v need not be induced; in fact, $N[S]$ need not induce a $K_{1,n}$.)

2.2 Application to computation of zero forcing number of families of graphs

In the preceding section, we obtained bounds on the power domination number using the zero forcing number. We take the opposite approach in this section, using Theorem 2.2 and results from power domination to obtain results for zero forcing.

In [8, Theorem 4.1] it was proved that:

$$\gamma_P(G*H) = \begin{cases} \gamma(G) & \text{if } \gamma_P(H)=1 \\ \gamma_t(G) & \text{otherwise,} \end{cases} \tag{2}$$

where $\gamma_t(G)$ denotes the *total domination number of* G, defined as the minimum cardinality of a dominating set S in G such that every vertex in S has at least one neighbor in S.

Now, from Theorem 2.2 we know $\mathrm{Z}(G*H) \le \gamma_P(G*H)\Delta(G*H)$. It follows easily from the definition of lexicographic product that $\deg_{G*H}(g,h) = (\deg_G g)|V(H)| + \deg_H h$ for any vertex $(g,h) \in V(G*H)$, and therefore $\Delta(G*H) = \Delta(G)|V(H)| + \Delta(H)$. Then from (2) above, we obtain

$$\mathrm{Z}(G*H) \le \begin{cases} \gamma(G)\left(\Delta(G)|V(H)| + \Delta(H)\right) & \text{if } \gamma_P(H)=1 \\ \gamma_t(G)\left(\Delta(G)|V(H)| + \Delta(H)\right) & \text{otherwise.} \end{cases} \tag{3}$$

In particular, we obtain the following result for lexicographic products of regular graphs with low domination and power dominations numbers.

Theorem 2.8. *Let G and H be regular graphs with degree d_G and d_H, respectively. If $\gamma_P(H) = 1$ and $\gamma(G) = 1$, then $\mathrm{Z}(G*H) = d_G|V(H)| + d_H$.*

Proof. Since G is d_G-regular, H is d_H-regular, and $\gamma_P(H) = \gamma(G) = 1$, the bound in (3) gives $\mathrm{Z}(G*H) \le d_G|V(H)| + d_H$. Moreover, since $G*H$ is $(d_G|V(H)|+d_H)$-regular, Observation 1.2 tells us $d_G|V(H)|+d_H = \delta(G*H) \le \mathrm{Z}(G*H)$. □

Corollary 2.9. *For $n \ge 2$ and $m \ge 3$, $\mathrm{Z}(K_n * C_m) = (n-1)m + 2$.*

3 Effect of graph operations on $\gamma_P(G)$

In this section we determine the effect of some graph operations on the power domination number by applying techniques from zero forcing. We utilize the fact that $S \subseteq V(G)$ is a power dominating set of G if and only if $N_G[S]$ is a zero forcing set for G, and repeatedly use the next observation.

Observation 3.1. *For any $S \subseteq V(G)$ and $v \in V(G)$, $N_G[S] \cup \{v\} \subseteq N_G[S \cup \{v\}]$, so if $N_G[S] \cup \{v\}$ is a zero forcing set for G, then $S \cup \{v\}$ is a power dominating set of G, and $\gamma_P(G) \le |S| + 1$.*

3.1 Vertex and edge deletion

The concept of rank spread was introduced in [3] to quantify the change in the minimum rank of a graph produced by the deletion of a vertex or an edge. In [10] this idea was extended to spreads of other parameters including the zero forcing number, with $z_v(G) = \mathrm{Z}(G) - \mathrm{Z}(G-v)$ and $z_e(G) = \mathrm{Z}(G) - \mathrm{Z}(G-e)$ where $G - e = (V, E \setminus \{e\})$. In both papers bounds on the spreads were proved. Similarly, we establish bounds on $\gamma_P(G) - \gamma_P(G-v)$ to quantify the effect that the deletion of a vertex (or analogously deletion of an edge) produces on the power domination number of a graph.

Proposition 3.2. *For every graph G and every vertex v, $\gamma_P(G) - 1 \le \gamma_P(G-v)$.*

Proof. Let T be a minimum power dominating set of $G-v$. Since $N_{G-v}[T]$ is a zero forcing set for $G-v$, $N_{G-v}[T] \cup \{v\}$ is clearly a zero forcing set for G, so by Observation 3.1, $\gamma_P(G) \le |T|+1 = \gamma_P(G-v)+1$. □

The next proposition shows there is no upper bound on $\gamma_P(G-v)$ in terms of $\gamma_P(G)$.

Proposition 3.3. *For every integer $r \geq -1$, there is a graph G_r with a vertex v such that $\gamma_P(G_r - v) = \gamma_P(G_r) + r$.*

Proof. If $r = -1$, it suffices to consider $G_{-1} = C_4 \circ K_1$ and a vertex v of degree 1 in G_{-1}, as shown in Figure 1. Then $\gamma_P(G_{-1} - v) = 1 = \gamma_P(G_{-1}) + (-1)$. Given an integer $n \geq 3$, $\gamma_P(C_n) = \gamma_P(P_{n-1}) = 1$, so $\gamma_P(C_n - v) = \gamma_P(C_n)$ for every vertex v in C_n. For any integer $r \geq 1$, consider the graph $G_r = K_{1,r+1}$ and the vertex v of maximum degree in G_r. Then $\gamma_P(G_r) = 1$ and $\gamma_P(G_r - v) = r + 1$. □

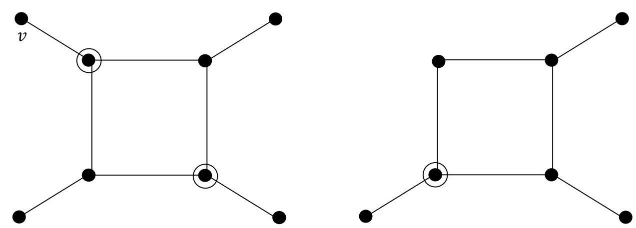

Figure 1: The graphs $G_{-1} = C_4 \circ K_1$ and $G_{-1} - v$ with vertices in their minimum power dominating sets circled

Proposition 3.4. *For every graph G and for every edge e, $\gamma_P(G) - 1 \leq \gamma_P(G - e) \leq \gamma_P(G) + 1$, and these bounds are tight.*

Proof. Assume $e = \{v, w\}$. For a zero forcing set B in one of G or $G - e$, we assume without loss of generality that v is blue at the time w is forced (or both v and w are in B). Then $B \cup \{w\}$ is a zero forcing set for the other of G or $G - e$, so by Observation 3.1, $\gamma_P(G - e) \leq \gamma_P(G) + 1$ and $\gamma_P(G) \leq \gamma_P(G - e) + 1$.

To see that the bounds on edge deletion are tight: Consider the graphs G_n, obtained from two disjoint copies of a cycle C_n by adding the edge $\{u, v\}$ where u and v are in different copies of the cycle, as shown in Figure 2. Then $\gamma_P(G_n) = 2$ and $\gamma_P(G_n - e) = 1$ for edge $e = \{v, w\}$ with $w \neq u$ (so e is in one of the cycles). Now consider the graphs H_n, obtained from two disjoint copies of a cycle C_n by adding respective edges between two adjacent vertices in each C_n, as shown in Figure 2; call those edges e and f. Then $\gamma_P(H_n) = 1$ and $\gamma_P(H_n - e) = 2$. There are also graphs for which deleting an edge does not alter the power domination number, such as the family of cycles C_n. □

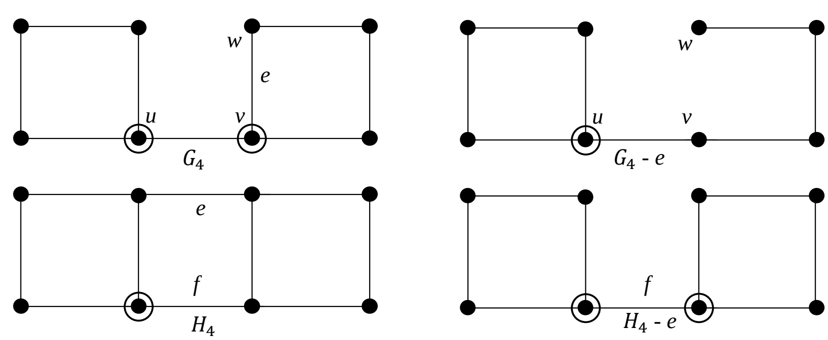

Figure 2: The graphs G_4, $G_4 - e$, H_4, and $H_4 - e$ with vertices in their minimum power dominating sets circled

3.2 Edge contraction

Let $e = \{u, w\}$ denote an arbitrary edge in a graph $G = (V, E)$. The *contraction* of G with respect to e, denoted by $G/e = (V^e, E^e)$, has $V^e = (V - \{u, w\}) \cup \{v\}$ and E^e is obtained from E by removing all edges incident with u or w (including e) and adding edges between the new vertex v and every vertex in G/e that is a neighbor in G of u or w. This process is also known as *identifying* the endpoints of e. The effect of edge contraction on the zero forcing number was determined in [14, Theorem 5.1].

Proposition 3.5. *For every graph G and for every edge e, $\gamma_P(G) - 1 \le \gamma_P(G/e) \le \gamma_P(G) + 1$, and these bounds are tight.*

Proof. Assume $e = \{u, w\}$ and v is the vertex resulting from identifying u and w. Since the inequality clearly holds for $G = K_2$ and γ_P sums across connected components, we can assume that $N_{G/e}(v) \neq \emptyset$. Also notice that $(G/e) - v = G - \{u, w\}$.

Let T be a minimum power dominating set of G/e. If $v \in T$, then $(T - \{v\}) \cup \{u, w\}$ is a power dominating set to G, so $\gamma_P(G) \le |T|+1 = \gamma_P(G/e)+1$. If $v \notin T$ and $v \in N_{G/e}(T)$, there exists a vertex $x \in T$ such that v is adjacent to x in G/e. Suppose, without loss of generality, that x is adjacent to u in G. Then, $N_G[T] \cup \{w\}$ is a zero forcing set for G, so $\gamma_P(G) \le |T| + 1 = \gamma_P(G/e) + 1$. Finally, suppose $v \notin N_{G/e}[T]$. Then some vertex x forces v. As before, suppose that x is adjacent to u in G. Then, as shown in the proof of [14, Theorem 5.1], $N_G[T] \cup \{w\}$ is a zero forcing set for G, which implies $\gamma_P(G) \le |T| + 1 = \gamma_P(G/e) + 1$. In all cases, $\gamma_P(G) \le \gamma_P(G/e) + 1$.

Let S be a minimum power dominating set of G. If at least one of $u, w \in S$, then since $N_{G/e}(v) = (N_G(u) \cup N_G(w)) \setminus \{u, w\}$ and no vertex of S performs a force, $(S \setminus \{u, w\}) \cup \{v\}$ is a power dominating set of G/e, implying $\gamma_P(G/e) \le |S| = \gamma_P(G)$. Now assume $u, w \notin S$. If at least one of u, w is in $N_G(S)$, then $N_{G/e}[S \cup \{v\}]$ contains the vertices (if any) that were forced by u and/or w and can therefore force G/e, so $S \cup \{v\}$ is a power dominating set of G/e, and $\gamma_P(G/e) \le |S| + 1 = \gamma_P(G) + 1$. Finally, suppose $v, w \notin N_G[S]$. Suppose u forces y before w performs a force (if neither performs a force, this step can be skipped and $N_{G/e}[S]$ is a zero forcing set). Then, as shown in the proof of [14, Theorem 5.1], $N_{G/e}[S] \cup \{y\}$ is a zero forcing set for G/e, which implies $\gamma_P(G/e) \le |S| + 1 = \gamma_P(G) + 1$. In all cases, $\gamma_P(G/e) \le \gamma_P(G) + 1$.

To see that the bounds on edge contraction are tight: Consider the graph G_n constructed from two disjoint copies of a cycle C_n by adding an edge e connecting the two copies of C_n. Then, as shown in Figure 3, $\gamma_P(G_n) = 2$ and $\gamma_P(G_n/e) = 1$. On the other hand, if H is the graph in Figure 3 with edge e as shown, then $\gamma_P(H) = 1$ and $\gamma_P(H/e) = 2$. □

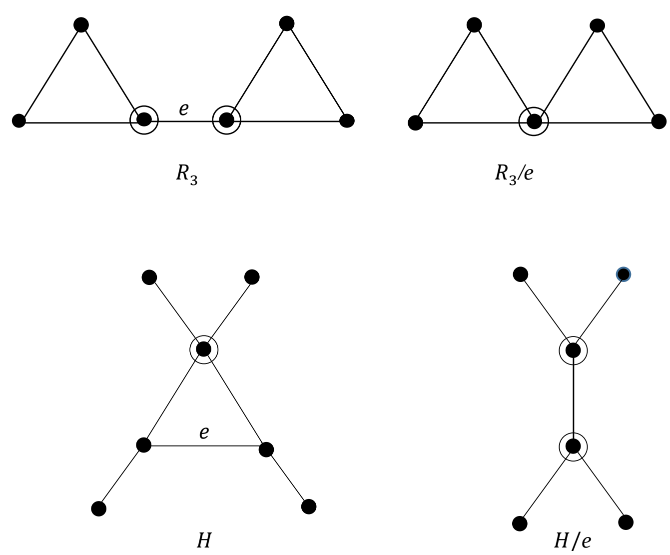

Figure 3: The graphs G_3, G_3/e, H, and and H/e with vertices in their minimum power dominating sets circled

3.3 Edge subdivision

Let $e = \{u, w\}$ denote an arbitrary edge in a graph $G = (V, E)$. The *subdivision graph* of G with respect to e, denoted by $G_e = (V_e, E_e)$, has $V_e = V \cup \{v_e\}$ and $E_e = (E \setminus \{e\}) \cup \{\{v_e, u\}, \{v_e, w\}\}$. That is, G_e is obtained by removing edge e from G and adding a new vertex v_e adjacent to the endpoints of e and to no other vertices.

Proposition 3.6. *For every graph G and every edge e, $\gamma_P(G) \leq \gamma_P(G_e) \leq \gamma_P(G) + 1$, and these bounds are tight.*

Proof. Assume $e = \{u, w\}$ and v_e is the vertex added in G_e. By Proposition 3.5 applied to G_e, $\gamma_P(G_e) - 1 \leq \gamma_P(G_e/\{u, v_e\})$. Since the graph resulting from the contraction of edge $\{u, v_e\}$ in G_e is isomorphic to G, $\gamma_P(G_e) \leq \gamma_P(G) + 1$.

Let S be a minimum power dominating set of G_e. If $v_e \in S$, then $(S \setminus \{v_e\}) \cup \{u\}$ is also a power dominating set of G_e, so without loss of generality assume $v_e \notin S$. We show S is a power dominating set of G. Consider a chronological list of forces $\mathcal{F}$ of $N_{G_e}[S]$ that colors G_e. One of four situations occurs (renaming u and w if necessary), depending on whether the forces $u \to v_e$ and/or $v_e \to w$ appear in $\mathcal{F}$. In case neither of the forces $u \to v_e$ nor $v_e \to w$ belongs to $\mathcal{F}$, then $\mathcal{F}$ is a chronological list of forces of $N_G[S]$ that colors G. In case both the forces $u \to v_e$ and $v_e \to w$ belong to $\mathcal{F}$, replace the pair of forces $u \to v_e$ and $v_e \to w$ by the force $u \to w$ to obtain a chronological list of forces of $N_G[S]$ that colors G. In case $u \to v_e$ belongs to $\mathcal{F}$ and $v_e \to w$ does not belong to $\mathcal{F}$, delete $u \to v_e$ to obtain a chronological list of forces of $N_G[S]$ that colors G. In case $u \to v_e$ does not belong to $\mathcal{F}$ and $v_e \to w$ does belong to $\mathcal{F}$, necessarily $v_e \in N_{G_e}[S]$, so $u \in S$, implying $u, w \in N_G[S]$. Deleting $v_e \to w$ gives a chronological list of forces of $N_G[S]$ that colors G.

To see that the bounds on edge subdivision are tight: If e is any edge in P_n, then $(P_n)_e = P_{n+1}$ and $\gamma_P(P_n) = \gamma_P(P_{n+1})$. Let G be the graph obtained from K_4 by subdividing each edge except e once, as shown in Figure 4. Then $\gamma_P(G) = 1$ and $\gamma_P(G_e) = 2$. □

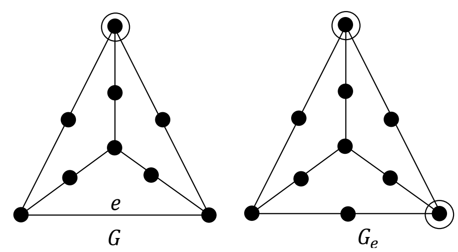

Figure 4: The graphs G and G_e with vertices in their minimum power dominating sets circled

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References

[1] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, and A. Wangsness). Zero forcing sets and the minimum rank of graphs. *Linear Algebra App.*, 428: 1628–1648, 2008.

[2] T.L. Baldwin, L. Mili, M.B. Boisen, Jr., and R. Adapa. Power system observability with minimal phasor measurement placement. *IEEE Trans. Power Syst.*, 8: 707–715, 1993.

[3] F. Barioli, S.M. Fallat, and L. Hogben. Computation of minimal rank and path cover number for graphs. *Linear Algebra App.*, 392: 289–303, 2004.

[4] R. Barrera and D. Ferrero. Power domination in cylinders, tori, and generalized Petersen graphs. *Networks*, 58: 43–49, 2011.

[5] G.J. Chang, P. Dorbec, M. Montassier, and A. Raspaud. Generalized power domination of graphs. *Discrete Appl. Math.*, 160: 1691–1698, 2012.

[6] L. DeAlba, J. Grout, L. Hogben, R. Mikkelson, and K. Rasmussen. Universally optimal matrices and field independence of the minimum rank of a graph. *Elec. J. Linear Algebra*, 18: 403–419, 2009.

[7] N. Dean, A. Ilic, I. Ramirez, J. Shen; K. Tian. On the Power Dominating Sets of Hypercubes. *IEEE 14th International Conference on Computational Science and Engineering (CSE)*, 488 - 491, 2011.

[8] P. Dorbec, M. Mollard, S. Klavžar, and S. Špacapan. Power domination in product graphs. *SIAM J. Discrete Math.*, 22: 554–567, 2008.

[9] M. Dorfling and M.A. Henning. A note on power domination in grid graphs. *Discrete Appl. Math.*, 154: 1023–1027, 2006.

[10] C.J. Edholm, L. Hogben, M. Huynh, J. LaGrange, and D.D. Row. Vertex and edge spread of zero forcing number, maximum nullity, and minimum rank of a graph. *Linear Algebra Appl.*, 436: 4352–4372, 2012.

[11] W.E. Ferguson, Jr. The Construction of Jacobi and Periodic Jacobi Matrices With Prescribed Spectra. *Math. Computation*, 35: 1203–1220, 1980.

[12] T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, and M.A. Henning. Domination in graphs applied to electric power networks. *SIAM J. Discrete Math.*, 15: 519–529, 2002.

[13] H.T. Hall. Spectra of cycles. Personal communication (notes available on request).

[14] K.D. Owens. Properties of the zero forcing number. Master's Thesis, Brigham Young University, 2009. Available at `http://scholarsarchive.byu.edu/cgi/viewcontent.cgi?article=3215&context=etd`.

[15] S. Stephen, B. Rajana, J. Ryan, C. Grigorious, and A. William. Power domination in certain chemical structures. *J. Discrete Algorithms*, 15: 519–529, 2002.