Product Line Sigraphs.

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Abstract

Intuitively, a signed graph is a graph in which every edge is labeled with a + or − sign. For each edge, its sign represents the mode of the relationship between the vertices it joins.

In a signed graph, cycles can be naturally given the sign corresponding to the product of its edges. Then, a signed graph is called balanced when all the cycles have positive sign. Balanced signed graph have multiple applications in the field of social networks. Consequently, there is a significant amount of research in the problem of determining if a signed graph is balanced or not. In particular, some authors investigated extensions to signed graph of the line graph and studied under what circumstances the signed graphs obtained are balanced or not.

This paper presents a new operation, which is also an extension to signed graphs of the line graph, with the property that applied to any signed graph always produces a balanced signed graph.

Keywords: *signed graphs, balance, line graph*

1. Introduction

Signed graphs or sigraphs, for short, are simple graphs (i.e. without loops or multiple edges) in which the edges are labeled with a sign, + or −. The motivation for studying sigraphs arises in the field of social sciences. For example, psychologists often model the sociometric structure of a group of people as a square matrix with elements −1, 0 and 1 representing disliking, indifference and liking, respectively. When a matrix of this sort is symmetric, it can be depicted by a signed graph whose positive edges indicate liking and the negative edges denote disliking.

In a sigraph G, a path of length p between two vertices u and v is a sequence of edges $P = a_0a_1, a_1a_2, \ldots, a_{p-1}a_p$ where $u = a_0$ and $a_p = v$. Then, the sign of the path P is defined as positive if the number of negative edges in P is even, and negative if the number of negative edges in P is odd. Precisely, $s(P) = \Pi_{i=0}^{p-1} s(a_ia_{i+1})$. Cycles represent a particular class of paths in which the end-vertices coincide. Therefore, cycles also have a corresponding sign. We say that a sigraph is balanced if and only if, all of its cycles are positive. Equivalently, Harary proved in [3] that a sigraph is balanced if and only if all paths between any two different vertices have the same sign.

In [1] and [2] the authors defined different operations on sigraphs, based on the line graph of an unsigned graph, and it was analyzed under what circumstances do those operations derive into balanced sigraphs. We propose an operation, also based on the notion of line graph of an unsigned graph, that always leads to a balanced sigraph.

For the completeness of this introduction, we recall the definition of line graph in the case of (unsigned) graphs. Let $G' = (V', E')$ be a graph, its line graph is denoted by $L(G')$, where the

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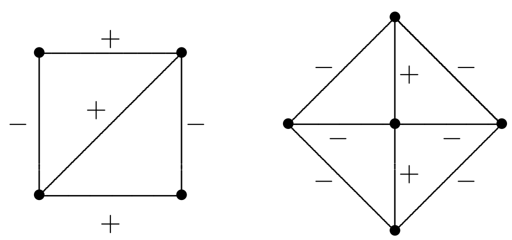
vertices of $L(G')$ correspond to the edges of G' and the edges of $L(G')$ represent edges in G' that share an endpoint.

We refer the reader to Zavlasky's dynamic survey on sigraphs [4] for additional concepts and for a description of the bibliography available in the field.

2. Product Line Sigraph

Definition 2.1 *Let $G = (V, E, s)$ be a sigraph, then the product line sigraph of G is defined as the sigraph $PL(G)$ whose vertices represent the edges of G. In $PL(G)$ two vertices are joined by a positive edge if and only if, they represent edges in G with the same sign. Analogously, two vertices of $PL(G)$ are joined by a negative edge if and only if, they represent edges in G with different signs.*



G $\qquad$ $PL(G)$

Figure 1. A sigraph G and its product line sigraph $PL(G)$.

Let $G = (V, E, s)$ be a sigraph. The *unsigned graph* of G is the graph $U(G) = (V, E)$. That is, the graph obtained by removing the signs from the edges of the signed graph.

Next we present a direct consequence from the definition of the product line sigraph. This property plays an important role in the study of balance in product line sigraphs.

Lemma 2.1 *For any sigraph G, the unsigned graph of its product line sigraph is isomorphic to the line graph of its unsigned graph. That is,*

$$U(PL(G)) \cong L(U(G)). \blacksquare$$

Analogous properties hold for other operations on sigraphs. Let G be a sigraph and let $L(G)$ be its line sigraph, as introduced and studied by Bezhad and Chartrand [2]. That is, the vertices of $L(G)$ represent the edges of G and two vertices of $L(G)$ are connected by an edge whenever the edges they represent form a path of length 2 in G. The sign of an edge in $L(G)$ is negative if and only if, it joins two vertices representing negative edges of G. Then, the property

$$U(L(G)) \cong L(U(G))$$

also holds. Indeed, this operation, as well as the product line sigraph, represents a different way of distributing signs to the edges of a line graph.

In the spirit of extending the notion of line graph, Acharya and Sinha [1] defined the common-edge sigraph. Let G be a sigraph, then the common-edge sigraph of G, denoted as $C(G)$, has for vertices the set of paths of length 2 in G. Two vertices in $C(G)$ are adjacent whenever the corresponding paths share an edge. The sign of an edge of $C(G)$ coincides with the sign of the edge shared by the paths its vertices represent. For common-edge sigraphs the relation with the line graph is given by

$$U(C(G)) \cong L^2(U(G)).$$

From this property and Lemma 2.1 we can conclude:

Lemma 2.2 *For any sigraph G, let $PL(G)$ denote its product line graph and let $C(G)$ denote the common-edge sigraph of G. Then,*

$$U(PL^2(G)) \cong U(C(G)).$$

Proof: If we apply Lemma 2.1 to $PL(G)$ we obtain that $U(PL^2(G)) = U(PL(PL(G))) \cong L(U(PL(G)))$ (1). Applying Lemma 2.1 again to G we obtain that $U(PL(G)) \cong L(U(G))$ and by applying the line graph to both sides of the previous congruence it results: $L(U(PL(G))) \cong L^2(U(G))$ (2). From the transitive property of the isomorphism, from (1) and (2) we conclude $U(PL^2(G)) \cong L^2(U(G))$, so $U(PL^2(G)) \cong U(C(G))$. $\blacksquare$

3. Balance

From this point forward, it is important to consider that we will exclusively deal with sigraphs G whose unsigned graph $U(G)$ has no loops or multiple edges.

Lemma 2.3 *Let G be a sigraph and let $PL(G)$ be its product line sigraph. Let Z be a cycle in $PL(G)$ such that there exists a cycle Z' in G satisfying $PL(Z') = Z$. Then Z is a positive cycle.*

Proof: Let us assume that the cycle Z' is formed by the union of consecutive paths $P_1, N_1, P_2, N_2, \dots P_k, N_k$, where the paths $P_1, \dots, P_k$ only contain positive edges and the paths $N_1, \dots, N_k$ only contain negative ones. Observe that for $i = 1, \dots, k$, in $PL(P_i)$ all the edges must be positive because there are no changes of sign in P_i. The same occurs with $PL(N_i)$. Therefore, the only negative edges in Z arise from the contact of edges in both ends of N_i with those end-edges of P_i and P_{i+1} for $i = 1, \dots, k-1$ or from the contact between N_k and both, P_k and P_0. In any case, each negative path N_i induces exactly two negative edges in Z, so the total number of negative edges will be even and hence, Z will be positive. ■

Intuitively, the previous proof is based on the fact that negative edges in $PL(G)$ arise from adjacent edges of G having different sign. Since we are working on a cycle, we start and end with the same sign so the number of changes in sign must be even, and therefore the cycle is positive. Still, to prove that $LP(G)$ is balanced we need to prove that all cycles in $PL(G)$ are positive, even those which do not arise from cycles in G. We will prove this statement by using the next two lemmas.

Lemma 2.4 *For any sigraph G, all triangles in $PL(G)$ have positive sign.*

Proof: Let $G = (V, E, s)$ be a sigraph and $LP(G) = (E, E', s')$ be its line sigraph. Since $U(PL(G)) \cong L(U(G))$, every triangle in $LP(G)$ is in unique correspondence with a triangle in $L(U(G))$, and therefore, it must arise either from a cycle C_3 or a star $K_{1,3}$ in the graph $U(G)$. Thus, given a a triangle in $LP(G)$, its sign will be determined by the signed copy of either C_3 or $K_{1,3}$ in G that leads to it. Therefore, these are all the possibilities we have:

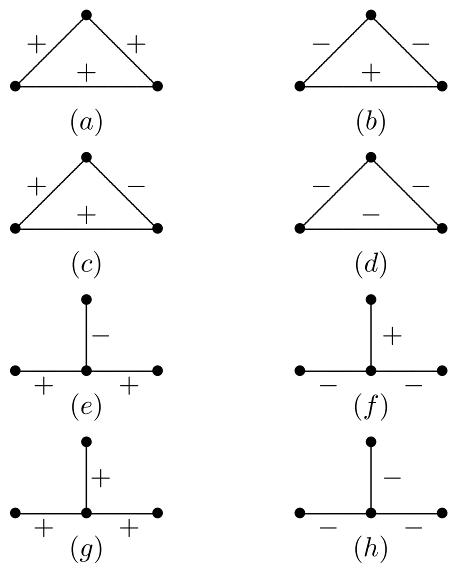


Figure 2. All possible sigraphs G isomorphic to C_3 or $K_{1,3}$.

Now, in the cases (a), (d), (g) and (h) of Figure 2 the product line sigraph is the same and coincides with the sigraph (i) of Figure 3, while in the cases (b), (f), (c) and (e) of Figure 2 the product line graph coincides with the sigraph (ii) of Figure 3.

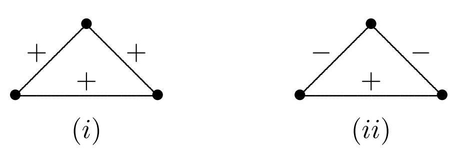


Figure 3. The sigraphs obtained from applying the product line sigraph to the sigraphs in Figure 1.

Since both sigraphs in Figure 3 are positive cycles, we conclude that all triangles in $LP(G)$

must be positive. ■

Theorem 2.5 *For any sigraph G, the product line sigraph $PL(G)$ is balanced.*

Proof: We need to prove that every cycle in $PL(G)$ has positive sign. Let Z be a cycle in $PL(G)$, since $U(G)$ has no loops or multiple edges then $U(PL(G))$ does not have loops or multiple edges and the length of Z must be at least 3. If the length of Z is 3, then Z is a triangle and Lemma 2.4 guarantees that its sign is positive. If the length of Z is greater than 3 we distinguish two cases. First, suppose that there exists a cycle Z' in G such that $Z = PL(Z')$. In this case Lemma 2.3 proves that Z must be positive. Second, suppose that there is no cycle in Z' in G such that $Z = PL(Z')$. In this case, there must be at least two adjacent edges in the cycle Z arising from a signed copy of $K_{1,3}$ in G. Let us assume that those edges in $PL(Z)$ join vertices corresponding to two edges e_p and e_q with e_t in G so that the sequence e_p, e_t, e_q represents a path in $LP(G)$ as it is shown in Figure 4. Since e_p, e_t and e_q arise from a copy of $K_{1,3}$ we will have the following structure:

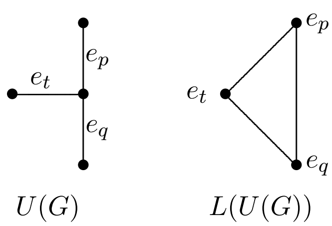


Figure 4. Structure in G that makes possible the construction of Z' from Z.

Then, e_p and e_q are also joined by an edge in $LP(G)$. By replacing the path e_p, e_t, e_q with the edge e_pe_q in the cycle Z we obtain a new cycle Z' of length one unit less than Z, where both Z and Z' have the same sign because all triangles are positive, so e_p, e_t, e_q has the same sign as e_pe_q. By repeating this procedure we can obtain a new cycle $Z*$ that either has length 3, or it has no edges arising from signed copies of $K_{1,3}$ in G. Since in this procedure the new cycle $Z*$ has the same sign as Z, and due to Lemmas 2.4 and Lemma 2.3 the sign of $Z*$ must be positive, we conclude that Z is positive. ■

Given a non-negative integer k and an arbitrary sigraph G, we recursively define $PL^k(G) = PL(PL^{k-1}(G))$ and $PL^0(G) = G$. The sigraph $PL^k(G)$ is called the k-iterated product line sigraph of G. Then, the following corollary holds:

Corollary 2.6 *For any sigraph G and for all non-negative integers k, the iterated product line sigraph $PL^k(G)$ is balanced.* ■

This corollary shows a property of the line sigraph that distinguishes it from the other operations mentioned before, since neither the line sigraph nor the common-edge sigraph preserve balance, as the following examples show.

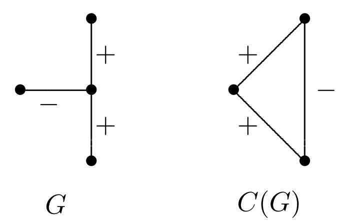


Figure 5. A balanced sigraph G such that $C(G)$ is not balanced.

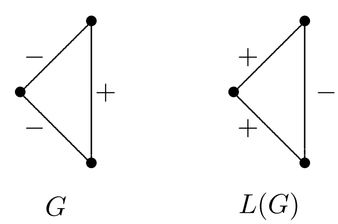


Figure 6. A balanced sigraph G such that $L(G)$ is not balanced.

4. Conclusion

As it was mentioned before, the line sigraph introduced by Behzad and Chartrand [2] and the common-edge sigraph introduced by Acharya and Sinha [1] also closely related to the line graph. However, in these operations it is not guaranteed that the sigraphs produced are balanced, as it happens with the product line graph. Furthermore, the iteration of the product line sigraph naturally produces balanced sigraphs, where the line sigraph and the common-edge sigraph do not preserve balance.

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