Some properties of superline digraphs.

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Abstract

For a given digraph $G = (V, E)$ and a positive integer k, the super line digraph of index k of G is the digraph $S_k(G)$ which has for vertices all the k-subsets of $E(G)$, and two vertices S and T are adjacent whenever there exist edges in the form $(u, v) \in S$ and $(v, w) \in T$ for some $u, v, w \in V$.

The super line digraph is a generalization of the super line graph. Indeed, if the digraph G is symmetric, the super line digraph of G is isomorphic to the super line graph of the graph obtained by removing the orientation of the edges of G. We study the link between properties of super line digraphs and super line graphs.

Keywords: *graphs, digraphs, line digraphs, super line graphs*

1. Introduction

In [3] Bagga, Beineke and Varma introduced the concept of super line graphs. For a given graph $G = (V, E)$ and a positive integer k, the super line graph of index k of G is the graph $S_k(G)$ which has for vertices all the k-subsets of $E(G)$, and two vertices S and T are adjacent whenever there exist $s \in S$ and $t \in T$ such that s and t share a common vertex. From the definition, it turns out that $S_1(G)$ coincides with the line graph $L(G)$. Properties of super line graphs were presented in [6], [4], [1] and [2], and a good and concise summary can be found in [11]. More specifically, some results regarding the super line graph of index 2 were presented in [5] and [1]. Several variations of the super line graph have been considered. For example, the super line graph of a multigraph was studied in [7]. In this paper we study the super line digraph which is defined as follows. For a given digraph $G = (V, E)$ and a positive integer k, the super line digraph of index k of G is the digraph $S_k(G)$ whose vertices are the k-subsets of $E(G)$, and two vertices S and T are adjacent whenever there exist edges in the form $(u, v) \in S$ and $(v, w) \in T$ for some $u, v, w \in V$.

Throughout the paper, $G = (V, E)$ is a digraph of order n vertices and size m edges. Let $V = \{v_1, \ldots, v_n\}$ be the set of vertices and $E = \{e_1, \ldots, e_m\}$ be the set of edges. By the definition of digraph, $E \in V \times V$. The digraph G can be associated with an adjacency matrix, which is the $n \times n$ matrix whose entries $a_{i,j}$ are given by $a_{i,j} = 1$ if there is an edge (v_i, v_j) in G and $a_{i,j} = 0$ otherwise.

We refer the reader to [9] and [10] for background on graph concepts not included in this Introduction.

2. Structural Properties

In [4] it was proved that if the number of components consisting of a single isolated edge of an undirected graph G is less than k, the superline graph of index k of G has diameter 1 or 2. We will now prove an analogous result for superline digraphs.

International Conference on Information Technology (ITNG'07)
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Lemma 2.1 *If S and T are two k-subsets of edges of a digraph G, such that neither of them contains only isolated edges. Then the distance from S to T in $S_k(G)$ is 1 or 2.*

Proof: If S is adjacent to T, then the distance from S to T in $S_k(G)$ is 1. If not, since neither of the set is composed by only isolated edges, there exist edges e_S and e_T such that e_S is adjacent from an edge in S and e_T is adjacent to an edge in T. Now, any k-subset of edges containing e_S and e_T will be adjacent from S and to T, forming a path of length 2 from S to T. ■

Theorem 2.2 *Let G be a digraph with less than k connected components consisting of a single isolated edge. Then, the diameter of $S_k(G)$ is 1 or 2.*

Proof: Since G has less than k connected components consisting of a single isolated edge, no k-subset of edges will be formed only by isolated edges. Therefore, using the previous lemma we obtain the result. ■

3. Completion Number

Observe that if for any two k-subsets of edges S and T there exist edges in the form $(u, v) \in S$ and $(v, w) \in T$ for some $u, v, w \in V$, then the same happens for any two $k+1$-subsets of edges. Therefore, it is clear that if every pair of vertices of $S_k(G)$ are adjacent, the same also holds in $S_{k+1}(G)$. This motivates the following definition. The completion number of a digraph G is $lc(G)$, the minimum integer k such that $S_k(G)$ is complete.

The following propositions show some bounds for the completion number of paths and cycles. Since their proofs are similar, we only present one of them.

Proposition 3.3 *Let P_n be the directed cycle of length n, i.e. with $n+1$ vertices. Then, $lc(P_n) \leq \lceil \frac{n}{2} \rceil$.* ■

Proposition 3.4 *Let C_n be the directed cycle in n vertices. Then, $lc(C_n) \leq \lceil \frac{n}{2} \rceil$.*

Proof: Let us label the vertices of the directed cycle C_n by $v_1, v_2, \ldots, v_n$, so that v_i is adjacent to v_{i+1} for all $n = 1, \ldots, n-1$ and v_n is adjacent to v_1. Now, for an edge (v_i, v_{i+1}), let us define $a(v_i, v_{i+1}) = (v_{i+1}, v_{i+2})$, for $n = 1, \ldots, n-2$ and $a(v_{n-1}, v_n) = (v_n, v_1)$, $a(v_n, v_1) = (v_1, v_2)$. Now, a vertex of $S_k(G)$ represents a subset of edges, let us say, $\{e_1, \ldots, e_k\}$. Then, if $k \geq \lceil \frac{n}{2} \rceil$, $|\{a(e_1), \ldots, a(e_k)\}| = k \geq \lceil \frac{n}{2} \rceil$. Therefore, $n - k < k$ and any subset of k edges of C_n will have at least one of the edges in $\{a(e_1), \ldots, a(e_k)\}$. This guarantees that the vertex $\{e_1, \ldots, e_k\}$ will be adjacent to every other vertex of $S_k(G)$. ■

Theorem 3.5 *Let G be a digraph with m edges and c components, then*

$$lc(G) \leq \lfloor \frac{m+c}{2} \rfloor$$

Moreover, this bound is sharp.

Proof: Let $k = \lfloor \frac{m+c}{2} \rfloor$ and let $l = lc(G)$. Let us assume that $S_k(G)$ is not complete. Then, there exist two vertices S and T such that S is not adjacent to T. As a consequence, $S \cap T$ must only contain isolated edges. Therefore, $k \leq \frac{m+c-2}{2}$, from which it follows that $l \leq \lceil \frac{m+c-1}{2} \rceil = \lfloor \frac{m+c}{2} \rfloor$.

Now let us show that this bound is sharp. Let us consider the family of cycles of even length, let us say C_{2p}, for all natural numbers p. Applying the previous Theorem we obtain $lc(C_{2p}) \leq \lfloor \frac{2p+1}{2} \rfloor = p$. On the other hand, by Proposition 3.4 we know that $lc(C_{2p}) = \lceil \frac{2p}{2} \rceil = p$. ■

4. Adjacency Matrix

Given two positive integers m and k where $m \geq k$, let $r = \binom{m}{k}$. We denote by $S_{m,k}$ the $r \times m$ binary matrix whose rows are the r strings with exactly k entries equal to 1. Then if G is a digraph with m edges, the rows of the matrix $S_{m,k}$ represent all the possible k-subsets of edges, or the vertices of the super line digraph $S_k(G)$.

Theorem 4.6 *If G is a digraph with m edges, for any integer k, $1 \leq k \leq m$, the adjacency matrix of the digraph $S_k(G)$ is*

$$A(S_k(G)) = S_{m,k}A(S_{m,k})^t$$

where $A = A(L(G))$ is the adjacency matrix of $L(G)$ and $(S_{m,k})^t$ denotes the transpose matrix of $(S_{m,k})$.

Proof: By definition of the product of matrices,

$$(S_{m,k}A(S_{m,k})^t)_{ij} = \sum_{p,q}(S_{m,k})_{ip}A_{pq}(S^t_{m,k})_{qj}$$

or equivalently,

$$(S_{m,k}A(S_{m,k})^t)_{ij} = \sum_{p,q}(S_{m,k})_{ip}A_{pq}(S_{m,k})_{jq}$$

For each pair of values for p and q, $(S_{m,k})_{ip}A_{pq}(S_{m,k})_{qj}$ is either 1 or 0. Moreover, $(S_{m,k})_{ip}A_{pq}(S_{m,k})_{qj} = 1$ exactly when $(S_{m,k})_{ip} = 1$, $A_{pq} = 1$ and $(S_{m,k})_{qj} = 1$. That is, when the vertex of $S_k(G)$ associated with row i in $S_{m,k}$ contains the edge e_p, the vertex of $S_k(G)$ associated with row j in $S_{m,k}$ contains the edge e_q, and e_p and e_q are adjacent edges in $L(G)$. Therefore, the sum expression for $(S_{m,k}A(S_{m,k})^t)_{ij}$ gives exactly the number of edges from the vertex i to the vertex j of $S_k(G)$. ■

The previous theorem relates the adjacency matrix of a super line digraph and that of the line digraph. Next we will obtain a relationship between the adjacency matrix of a digraph and that of a super line digraph. For that purpose we use the following terminology, introduced first in [8].

If $G = (V, E)$ is a digraph with vertices $V = \{v_1, \ldots, v_n\}$ and edges $E = \{e_1, \ldots, e_m\}$. For an edge $e = (u, v)$ we define the head of e as $h(e) = u$ and the tail of e as $t(e) = v$. Now, the incidence matrix of heads of the digraph G has entries

$$h_{i,j} = \begin{cases} 1, \text{ if } h(e_j) = v_1; \\ 0, \text{ otherwise }. \end{cases}$$

Analogously, the incidence matrix of tails of the digraph G has entries

$$t_{i,j} = \begin{cases} 1, \text{ if } t(e_j) = v_1; \\ 0, \text{ otherwise }. \end{cases}$$

Proposition 4.7 *[8] If G is a digraph with incidence matrix of heads H and incidence matrix of tails T. Then, the adjacency matrix of $L(G)$ is given by T^tH.*

Now, using Proposition 4.7, we can replace A with T^tH in Theorem 4.6 and obtain the following corollary.

Theorem 4.8 *If G is a digraph with m edges, for any integer k, $1 \leq k \leq m$, the adjacency matrix of the digraph $S_k(G)$ is*

$$A(S_k(G)) = S_{m,k}T^tH(S_{m,k})^t$$

where H is the incidence matrix of heads and T is the incidence matrix of tails of G. ■

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