The structure of super line graphs.

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Abstract

For a given graph $G = (V, E)$ and a positive integer k, the super line graph of index k of G is the graph $S_k(G)$ which has for vertices all the k-subsets of $E(G)$, and two vertices S and T are adjacent whenever there exist $s \in S$ and $t \in T$ such that s and t share a common vertex. In the super line multigraph $L_k(G)$ we have an adjacency for each such occurrence.

We give a formula to find the adjacency matrix of $L_k(G)$. If G is a regular graph, we calculate all the eigenvalues of $L_k(G)$ and their multiplicities. From those results we give an upper bound on the number of isolated vertices.

1. Introduction

In [4] Bagga, Beineke and Varma introduced the concept of super line graphs. For a given graph $G = (V, E)$ and a positive integer k, the *super line graph of index k* of G is the graph $S_k(G)$ which has for vertices all the k-subsets of $E(G)$, and two vertices S and T are adjacent whenever there exist $s \in S$ and $t \in T$ such that s and t share a common vertex. From the definition, it turns out that $S_1(G)$ coincides with the line graph $L(G)$. Properties of super line graphs were presented in [7], [5] and [2], and a good and concise summary can be found in [12]. More specifically, some results regarding the super line graph of index 2 were presented in [6] and [2]. Several variations of the super line graph have been considered. A recent survey of line graphs and their generalizations is given in [1].

In this paper we study the super line multigraph which is defined as follows. For a given graph $G = (V, E)$ and a positive integer k, the *super line multigraph of index k* of G is the multigraph $L_k(G)$ whose vertices are the k-subsets of $E(G)$, and two vertices S and T are joined by as many edges as pairs of edges $s \in S$ and $t \in T$ share a common vertex.

A graph G on n vertices $\{v_1, \ldots, v_n\}$ can be associated with an *adjacency matrix*, which is the $n \times n$ matrix whose entries $a_{i,j}$ are given by $a_{i,j} = 1$ if there is an edge joining v_i and v_j in G and $a_{i,j} = 0$ otherwise. For a multigraph, $a_{i,j}$ is the number of edges between v_i and v_j. The *characteristic polynomial* of the graph G, denoted as $\chi(G, \lambda)$, is defined as $\det(A - \lambda I)$. The *eigenvalues* of the graph G are those of A. The *algebraic multiplicity* of an eigenvalue α is the multiplicity of α as a root of the characteristic polynomial of A and is denoted as $m_a(\alpha, A)$ or $m_a(\alpha, G)$. The *geometric multiplicity* of an eigenvalue α is the dimension of $ker(A - \alpha I)$ and is denoted as $m_g(\alpha, A)$ or $m_g(\alpha, G)$. If A is a real symmetric matrix, every eigenvalue α satisfies $m_a(\alpha, A) = m_g(\alpha, A)$. The *spectrum* of G is the set of eigenvalues of G together with their multiplicities as eigenvalues of A. The spectrum of a graph provides valuable information on its topology. More information on this topic can be found in [8] and [11].

We refer the reader to [9] and [10] for background on graph concepts not included in this Introduction.

2. Spectral properties

Given two positive integers n and k where $n \geq k$, let $m = \binom{n}{k}$. We denote by $S_{n,k}$ the $m \times n$ binary matrix whose rows are the m strings with exactly k entries equal to 1. Then if G is a graph with n edges, the rows of the matrix $S_{n,k}$ represent all the possible k-subsets of edges, or the vertices of the super line multigraph $L_k(G)$.

Lemma 2.1 *If G is a graph with n edges, for any integer k, $1 \leq k \leq n$, the adjacency matrix of the multigraph $L_k(G)$ is*

$$A(L_k(G)) = S_{n,k} A (S_{n,k})^t$$

where $A = A(L(G))$ is the adjacency matrix of $L(G)$. ■

Corollary 2.2 *If G is a graph with n edges, for any integer k, $1 \leq k \leq n$, the adjacency matrix of the multigraph*

Proceedings of the 8th International Symposium on Parallel Architectures, Algorithms and Networks (ISPAN'05)
1087-4089/05 $20.00 © 2005 IEEE

$L_k(G)$ *is*

$$A(L_k(G)) = S_{n,k}B^tB(S_{n,k})^t - 2S_{n,k}(S_{n,k})^t$$

where $B = B(L(G))$ *is the incidence matrix of* G. ■

Notice that if $k > |E|$ then $S_k(G)$ has no vertices and if $k = |E|$, $S_k(G)$ is a trivial graph with one vertex. For this reason, we focus our attention to the case $1 \leq k < |E|$.

Proposition 2.3 *Let* G *be a graph with* n *edges,* k *an integer,* $1 \leq k < n$*, and* A *the adjacency matrix of* $L(G)$*. Let* $m = \binom{n}{k}$*; then*

$$\chi(S_{n,k}AS_{n,k}{}^t;\lambda) = (-\lambda)^{m-n}\chi(S_{n,k}{}^tS_{n,k}A;\lambda).$$

Proof: For any eigenvalue α of $S_{n,k}AS_{n,k}^t$, define $r_\alpha = m_a(\alpha, S_{n,k}AS_{n,k}^t)$. We first show that for eigenvalues $\alpha \neq 0$ of $S_{n,k}AS_{n,k}^t$,

$$\begin{aligned} r_\alpha &= m_g(\alpha, S_{n,k}AS_{n,k}^t) \\ &\leq m_g(\alpha, S_{n,k}^tS_{n,k}A) \\ &\leq m_a(\alpha, S_{n,k}^tS_{n,k}A) \end{aligned}$$

The first equality is clear since $S_{n,k}AS_{n,k}^t$ is real and symmetric. Suppose $\alpha \neq 0$ is an eigenvalue of $S_{n,k}AS_{n,k}^t$. Let $v_1, \ldots, v_{r_\alpha}$ be a basis for the eigenspace corresponding to α. For any nonzero linear combination v of $v_1, \ldots, v_{r_\alpha}$, we must have $S_{n,k}^tv \neq 0$ and $S_{n,k}^tS_{n,k}AS_{n,k}^tv = \alpha S_{n,k}^tv$. Therefore $m_g(\alpha, S_{n,k}AS_{n,k}^t) \leq m_g(\alpha, S_{n,k}^tS_{n,k}A)$ and $(\lambda-\alpha)^{r_\alpha}|\chi(S_{n,k}^tS_{n,k}A;\lambda)$.

Now consider $\alpha = 0$, let E_0 be the eigenspace of $S_{n,k}AS_{n,k}^t$ corresponding to eigenvalue 0, and let $N(S_{n,k}^t)$ be the null space of $S_{n,k}^t$. The rank of $S_{n,k}^t$ is n, since it is easy to see that the n linearly independent n-vectors $e_i + e_n$, for $1 \leq i \leq n-1$, and $e_1 + \cdots + e_n$ are in its column space. Therefore the dimension of $N(S_{n,k}^t)$ is $m-n$. Clearly, $N(S_{n,k}^t) \subseteq E_0$. Let $v_1, \ldots, v_{m_N}, v_{m-n}, \ldots, v_{r_0}$ be a basis for E_0 extended from $v_1, \ldots, v_{m-n}$, a basis for $N(S_{n,k}^t)$. Thus any nonzero linear combination v of $v_{m-n+1}, \ldots, v_{r_0}$ yields a nonzero eigenvector $S_{n,k}^tv$ of $S_{n,k}^tS_{n,k}A$, giving that $\lambda^{r_0-m+n}|\chi(S_{n,k}^tS_{n,k}A;\lambda)$. Therefore $\chi(S_{n,k}AS_{n,k}^t;\lambda)|\lambda^{m-n}\chi(S_{n,k}^tS_{n,k}A;\lambda)$, and we have equality up to a sign because the polynomials are monic and the degrees are equal. ■

Note that if one follows the definition of characteristic polynomial given in [8] the above Proposition implies that $\chi(S_{n,k}AS_{n,k}{}^t;\lambda) = \lambda^m\chi(S_{n,k}{}^tS_{n,k}A;\lambda)$.

Observe that $m-n$ is positive, unless $k=1$ or $k=n-1$.

Obviously, $S_{n,1}{}^tS_{n,1} = I$. However, if $k > 1$, $S_{n,k}{}^tS_{n,k}$ still has a particular pattern. In what follows, for any matrix X, let us denote by $(X)_{ij}$ the entry in the row i and column j in the matrix X.

Lemma 2.4 *For an integer* $n \geq 2$ *and an integer* k*,* $1 \leq k < n$*, let* b *and* c *be defined by*

$$b = \binom{n-2}{k-2} \text{ and } c = \binom{n-1}{k-1}$$

Then

$$S_{n,k}{}^tS_{n,k} = bJ + (c-b)I,$$

where J *is the all-one's matrix.*

Proof: $S_{n,k}{}^tS_{n,k}$ is a $n \times n$ matrix whose entries are

$$\begin{aligned} (S_{n,k}{}^tS_{n,k})_{ij} &= \textstyle\sum_{k=1}^{\binom{n}{k}}(S_{n,k})_{ik}^t(S_{n,k})_{kj} \\ &= \textstyle\sum_{k=1}^{\binom{n}{k}}(S_{n,k})_{ki}(S_{n,k})_{kj} \end{aligned}$$

Therefore if $i = j$, $(S_{n,k})_{ki}$ and $(S_{n,k})_{kj}$ coincide in exactly $\binom{n-1}{k-1}$ positions, and if $i \neq j$, in exactly $\binom{n-2}{k-2}$ positions. Thus the entries of the matrix $S_{n,k}{}^tS_{n,k}$ are all $b = \binom{n-2}{k-2}$, except in the diagonal where all are $c = \binom{n-1}{k-1}$. ■

From Lemma 2.4 we shall obtain $\chi(S_{n,k}{}^tS_{n,k}A;\lambda)$, and therefore $\chi(S_{n,n-1}AS_{n,n-1}{}^t;\lambda)$, for regular graphs.

3. Regular graphs

In this section we consider a d-regular graph G with n edges which has $2(d-1)$-regular line graph $L(G)$. If A is the adjacency matrix of $L(G)$, then A has eigenvalues $\alpha_1 = 2(d-1), \alpha_2, \ldots, \alpha_n$ with corresponding eigenvectors $\phi_1 = \vec{1}, \phi_2, \ldots, \phi_n$.

Proposition 3.1 *Let* G *be a* d*-regular graph with* $n \geq 2$ *edges,* k *an integer,* $1 \leq k < n$*, and* A *the adjacency matrix of* $L(G)$*. Let* $\alpha_1, \ldots, \alpha_n$ *be the eigenvalues of* A *and* ϕ_i *an eigenvector corresponding to the eigenvalue* α_i*, for* $i = 1, \ldots, n$ *where* $\phi_1 = \vec{1}$*, the all 1's vector. Then the eigenvalues of* $S_{n,k}^tS_{n,k}A$ *are* $\lambda_1 = 2bn(d-1) + 2(c-b)(d-1)$ *with eigenvector* $\phi_1 = \vec{1}$*, and* $\lambda_i = (c-b)\alpha_i$*, for* $i = 2, \ldots, n$*, with eigenvector* ϕ_i*. Therefore*

$$\chi(S_{n,k}{}^tS_{n,k}A;\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)$$

Proof: From Lemma 2.4, ${S_{n,k}}^t S_{n,k} = bJ + (c-b)I$, where $b = \binom{n-2}{k-2}$ and $c = \binom{n-1}{k-1}$. Therefore, since $L(G)$ is $2(d-1)$-regular, $S^t_{n,k} S_{n,k} A = 2b(d-1)J + (c-b)A$. Besides, because $L(G)$ is $2(d-1)$-regular, A and J have eigenvector $\vec{1}$ simultaneously, and so all other eigenvectors of A are also eigenvectors of J. All other eigenvectors of A are in the null space of J. ■

Theorem 3.2 *Let G be a d-regular graph with n edges, k an integer, $1 \leq k < n$, and A the adjacency matrix of $L(G)$. Let $m = \binom{n}{k}$. Then*

$$\chi(S_{n,k} A {S_{n,k}}^t; \lambda) = \pm\lambda^m \prod_{i=1}^{n} (\lambda - \lambda_i)$$

for $\lambda_1 = 2bn(d-1) + 2(c-b)(d-1)$ and $\lambda_i = (c-b)\alpha_i$, $i = 2, \ldots, n$. ■

The previous theorem is a direct consequence of Lemma 2.3 and Proposition 3.1. It gives the spectrum of the super-k multigraph $L_k(G)$, in the case where G is d-regular and $2 \leq k \leq n$.

4. Independence number

The *independence number* of a graph G, denoted as $\alpha(G)$, is the size of a maximal independent set of vertices (i.e. a set of vertices whose induced subgraph has no edges). Some results regarding independence in super line graphs are presented in [3].

From the adjacency matrix of a super line multigraph $L_k(G)$ we shall obtain upper bounds for the independence number of the super line graph $S_k(G)$, using the following result found in [11, p. 205].

Lemma A. Let X be a graph on n vertices and let A be a symmetric $n \times n$ matrix such that $A_{uv} = 0$ if the vertices u and v are not adjacent. Then

$$\alpha(X) \leq \min\left\{n - n^+(A), n - n^-(A)\right\}.$$

where $n^+(A)$ and $n^-(A)$ denote the number of positive and negative eigenvalues of A, respectively. ■

Since non-adjacent vertices in $S_k(G)$ are also non-adjacent in $L_k(G)$ we can apply the previous result to the super line graph $S_k(G)$ and the adjacency matrix of the super line multigraph, $A(L_k(G))$.

Proposition 4.1 *Let G be a d-regular graph with n edges, k an integer, $1 \leq k < n$. Then $\alpha(S_k(G)) \leq$ is at most*

$$\min\left\{\binom{n}{k} - n^+(L(G)), \binom{n}{k} - n^-(L(G))\right\}$$

where $n^+(L(G))$ and $n^-(L(G))$ denote the number of positive and negative eigenvalues of $L(G)$, respectively.

Proof: From Lemma A, $\alpha(S_k(G)) \leq \min\{\binom{n}{k} - n^+(L_k(G)), \binom{n}{k} - n^-(L_k(G))\}$, where $n^+(L_k(G))$ and $n^-(L_k(G))$ denote the number of positive and negative eigenvalues of $L_k(G)$, respectively. From Proposition 3.1 it is easy to see that $n^+(L_k(G)) = n^+(L(G))$ and $n^-(L_k(G)) = n^-(L(G))$, which proves the inequality for $\alpha(S_k(G))$. ■

In [7] was proved that a super line graph is almost connected (i.e. at most one connected component contains edges). From the above result, it is obvious that $\min\{\binom{n}{k} - n^+(L(G)), \binom{n}{k} - n^-(L(G))\} - 1$ gives an upper bound for the number of isolated vertices in a super line graph or multigraph.

References

[1] Jay Bagga. Old and New Generalizations of Line Graphs. International Journal of Mathematics and Mathematical Sciences 29 (2004), 1509-1521.

[2] K. S. Bagga, L. W. Beineke, B. N. Varma. The super line graph L_2. Discrete Math. 206 (1999), no. 1-3, 51–61.

[3] K. S. Bagga, L. W. Beineke, B. N. Varma. Independence and cycles in super line graphs. Australas. J. Combin. 19 (1999), 171–178.

[4] K. S. Bagga, L. W. Beineke, B. N. Varma. Super line graphs. Graph theory, combinatorics, and algorithms, Vol. 1, 2 (Kalamazoo, MI, 1992), 35–46, Wiley-Intersci. Publ., Wiley, New York, 1995.

[5] K. S. Bagga, L. W. Beineke, B. N. Varma. Super line graphs and their properties. Combinatorics, graph theory, algorithms and applications (Beijing, 1993), 1–6, World Sci. Publishing, River Edge, NJ, 1994.

[6] K. S. Bagga, M. R. Vasquez. The super line graph L_2 for hypercubes. Congr. Numer. 93 (1993), 111–113.

[7] K. S. Bagga, L. W. Beineke, B. N. Varma. The line completion number of a graph. Graph theory, combinatorics, and algorithms, Vol. 1, 2 (Kalamazoo, MI, 1992), 1197–1201, Wiley-Intersci. Publ., Wiley, New York, 1995.

[8] N. Biggs. *Algebraic Graph Theory*. Cambridge University Press (1974).

[9] G. Chartrand, L. Lesniak. *Graphs and Digraphs*. Chapman and Hall (1996).

[10] F. Harary. *Graph Theory*. Addison-Wesley, Reading MA (1969).

[11] C. Godsil, G. Royle. *Algebraic Graph Theory*. Graduate Texts in Mathematics. Springer (2001).

[12] E. Prisner. *Graph Dynamics*. Pitman Research Notes in Mathematics Series. Longman (1995).

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