Diameter Vulnerability of Iterated Line Digraphs in Terms of the Girth

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Iterated line digraphs arise naturally in designing fault tolerant systems. Diameter vulnerability measures the increase in diameter of a digraph when some of its vertices or arcs fail. Thus, the study of diameter vulnerability is a suitable approach to the fault tolerance of a network. In this article we present some upper bounds for diameter vulnerability of iterated line digraphs L^kG. Our bounds depend basically on the girth of the digraph G and on the number of iterations k. These bounds generalize some previous results on diameter vulnerability of line digraphs. Also, we apply our results to several important families of line digraphs such as Kautz digraphs and deBruijn generalized cycles, which contain deBruijn digraphs, the Reddy-Pradhan-Kuhl digraphs, and the butterflies. Our bounds allow us to obtain improvements in known results on diameter vulnerability for all these families. **NETWORKS, Vol. 45(2), 49–54 2005**

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1. INTRODUCTION

Graphs and digraphs are a useful tool to model communication networks, because nodes and links can be naturally represented by vertices and edges. As a consequence, several interesting problems concerning interconnection networks may be solved by studying different properties of these structures.

This work deals with digraphs and a problem related to fault tolerance. More specifically, if some nodes or links of a network cease to function, it is desirable that the remaining nodes can efficiently communicate. Thus, it is particularly interesting to study the behavior of some parameters related to the efficiency of a network in the presence of faults. As a consequence, diameter vulnerability has been considered, because it provides a measure of the maximum communication delay between any two nodes [3, 4]. Most of the known results on this topic are bounds for particular families of digraphs. The starting point in this work will be the results for general iterated line digraphs obtained in [10, 18]. Taking into account the girth of the original digraph we obtain new bounds on diameter vulnerability of line digraphs, which represent an improvement in the results given in [18]. Finally, we apply our results to several important families of line digraphs such as Kautz digraphs, and deBruijn generalized cycles, which contain the deBruijn digraphs [5], the Reddy-Pradhan-Kuhl digraphs [20], and the butterflies [1]. All of these families have been considered as very important interconnection network models. Our bounds allow us to improve almost all known results concerning their diameter vulnerability.

This article is organized as follows. In Section 2 we give some definitions and notation that will be used throughout the article. In Section 3 we show the role of the girth in relation to the diameter vulnerability of iterated line digraphs. Finally, in Section 4, we apply our results to some important families contained in the class of iterated line digraphs.

2. DEFINITIONS AND NOTATION

We recall some basic concepts and terminology and refer the reader to [6] for additional graph concepts.

From now on, G stands for a simple digraph, that is one without loops or multiple arcs, with the set of vertices $V(G)$ and the set of arcs $A(G)$. If $x \in V(G)$, let $\Gamma^-(x)$ and $\Gamma^+(x)$ denote, respectively, the sets of vertices adjacent to and from x. The *minimum degree* of G will be denoted by $\delta = \delta(G)$. A path from a vertex x to a vertex y will be referred

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to as an $x \to y$ path. The *distance from x to y* is denoted by $d_G(x,y) = d(x,y)$, and given a subset of vertices F, the distance from x to F is defined as $d(x,F) = \min\{d(x,y){:}y \in F\}$. The distance $d(F,x)$ is defined analogously. The *diameter* of G is $D = D(G) = \max_{x,y \in V(G)}\{d(x,y)\}$. The *girth* of G is usually denoted by g, and it is defined as the length of the shortest cycle.

A digraph G is said to be (*strongly*) *connected* when for any pair of vertices $x, y \in V(G)$ there exists an $x \to y$ path. Throughout this article we deal with a connected digraph G. As usual, the *connectivity* (or *vertex-connectivity*) and *edge-connectivity* of G are denoted by $\kappa = \kappa(G)$, and $\lambda = \lambda(G)$, respectively. It is well known that $\kappa \leq \lambda \leq \delta$ [14]. Hence, G is said to be *maximally connected* when $\kappa = \lambda = \delta$, and *maximally edge-connected* when $\lambda = \delta$.

In the *line digraph* LG of a digraph G each vertex represents an arc of G, that is, $V(LG) = \{uv : (u,v) \in A(G)\}$. A vertex uv is adjacent to a vertex wz if $v = w$, that is, whenever the arc (u,v) of G is adjacent to the arc (w,z). The line digraph operation is a useful tool for constructing large digraphs with fixed degree and small diameter. If G is a digraph different from a cycle, with minimum degree δ and diameter D, then the iterated line digraph L^kG has minimum degree δ and diameter $D + k$. The set of vertices of L^kG can be seen as the set of all walks of length k in G, represented by sequences $x_0x_1 \dots x_k$, where (x_i, x_{i+1}) is an arc of G. A vertex $\mathbf{x} = x_0x_1 \dots x_k$ in L^kG is adjacent to the vertex $\mathbf{y} = x_1 \dots x_k x_{k+1}$ for all x_{k+1} adjacent from x_k. A path of length h in L^kG can be written as a sequence of $k + h + 1$ vertices of G, where the vertices of this path are the subsequences of $k + 1$ consecutive vertices of G. Notice that between any pair of vertices of L^kG there exists at most one path of length less than or equal to $k + 1$. See [13] for proofs and more information.

The parameter ℓ [8, 11] appears to be very useful in the study of properties related to the connectivity. Let G be a digraph with minimum degree δ and diameter D. Let $\ell = \ell(G)$, $1 \leq \ell \leq D$, be the greatest integer such that, for any $x, y \in V(G)$,

1. if $d(x,y) < \ell$, the shortest $x \to y$ path is unique, and there are no paths of length $d(x,y) + 1$;
2. if $d(x,y) = \ell$, there is only one shortest $x \to y$ path.

Note that $\ell \geq 1$. It was also proven in [8] that $\ell(L^kG) = \ell + k$ if G has minimum degree $\delta \geq 2$. Also, in [8, 11] it was proven that $\kappa = \delta$ if $D \leq 2\ell - 1$, and $\lambda = \delta$ if $D \leq 2\ell$. As a consequence,

$$\begin{aligned} \kappa(L^kG) &= \delta \quad \text{if } k \geq D - 2\ell + 1; \\ \lambda(L^kG) &= \delta \quad \text{if } k \geq D - 2\ell. \end{aligned} \tag{1}$$

For a digraph G and a positive integer s, the *s-vertex diameter-vulnerability* $K(s,G)$ is the maximum of the diameters of the digraphs obtained by removing s arbitrary vertices of G. Analogously can be defined the *s-arc diameter-vulnerability* denoted by $\Lambda(s,G)$. From the definition, $K(0,G)$ and $\Lambda(0,G)$ coincide with the diameter D of G. The connectivities $\kappa(G)$ and $\lambda(G)$ are, respectively, the minimum values of s satisfying $K(s,G) = \infty$ and $\Lambda(s,G) = \infty$.

3. DIAMETER VULNERABILITY OF ITERATED LINE DIGRAPHS

In this section we improve the results given in [18] by taking into account the girth. First, we prove some useful results concerning cycles in L^kG. More precisely, in [18] it is shown that for any given two cycles of L^kG, $\mathbf{x}\mathbf{v}_1 \dots \mathbf{v}_{m_1-1}\mathbf{x}$ and $\mathbf{x}\mathbf{w}_1 \dots \mathbf{w}_{m_2-1}\mathbf{x}$, such that $\mathbf{v}_1 \neq \mathbf{w}_1$, if the digraph has loops ($g = 1$) then $m_1 + m_2 \geq k + 3$, and if not, $m_1 + m_2 \geq k + 4$. The next lemma generalizes such result.

Lemma 3.1. *Let G be a digraph with girth $g \geq 2$ and let $\mathbf{x}$ be a vertex of L^kG, $k \geq 0$. Let $\mathbf{x}\mathbf{v}_1 \dots \mathbf{v}_{m_1-1}\mathbf{x}$ and $\mathbf{x}\mathbf{w}_1 \dots \mathbf{w}_{m_2-1}\mathbf{x}$ be two cycles in L^kG. If $\mathbf{v}_1 \neq \mathbf{w}_1$, the sum of the lengths of the two cycles is at least $k + g + 2$. That is, $m_1 + m_2 \geq k + g + 2$.*

Proof. Observe that the girth of L^kG is g for every $k \geq 0$. Notice that if either $m_1 \geq k + 2$ or $m_2 \geq k + 2$ the result is obvious. Then, let us assume that $g \leq m_1 \leq m_2 \leq k + 1$ and $m_1 + m_2 \leq k + g + 1$ (otherwise, we would have finished). Let $\mathbf{x} = x_0x_1 \dots x_k$, where the x_i are vertices of G. The cycles $C_1 = \mathbf{x}\mathbf{v}_1 \dots \mathbf{v}_{m_1-1}\mathbf{x}$ and $C_2 = \mathbf{x}\mathbf{w}_1 \dots \mathbf{w}_{m_2-1}\mathbf{x}$ can be respectively denoted by the sequences of vertices of G, $x_0x_1 \dots x_k y_0 y_1 \dots y_{m_1-1}$ and $x_0x_1 \dots x_k z_0 z_1 \dots z_{m_2-1}$, where $y_0 \neq z_0$ because $\mathbf{v}_1 \neq \mathbf{w}_1$. Because the vertex $\mathbf{x}$ appears in the m_1-th position of the cycle C_1, then $\mathbf{x} = x_0x_1 \dots x_k = x_{m_1} \dots x_k y_0 y_1 \dots y_{m_1-1}$. When $m_1 \leq k$, looking at this equality term by term, we obtain the system of equations: $x_0 = x_{m_1}, x_1 = x_{m_1+1}, \dots, x_{k-m_1} = x_k$, $x_{k-m_1+1} = y_0, \dots, x_k = y_{m_1-1}$. That is, when $m_1 \leq k$, the following condition is satisfied:

$$\text{if } i \equiv j \pmod{m_1}, \text{ then } x_i = x_j \text{ for all } 0 \leq i,j \leq k. \tag{2}$$

Analogously, as the vertex $\mathbf{x}$ appears in the m_2-th position at the cycle C_2, then $\mathbf{x} = x_0x_1 \dots x_k = x_{m_2} \dots x_k z_0 z_1 \dots z_{m_2-1}$. Hence, if $m_2 \leq k$, we have the system of equations: $x_0 = x_{m_2}$, $x_1 = x_{m_2+1}, \dots, x_{k-m_2} = x_k$, $x_{k-m_2+1} = z_0, \dots, x_k = z_{m_2-1}$. That is, the following condition is also satisfied when $m_2 \leq k$:

$$\text{if } i \equiv j \pmod{m_2}, \text{ then } x_i = x_j \text{ for all } 0 \leq i,j \leq k. \tag{3}$$

We consider an integer R, $0 \leq R \leq m_1 - 1$, such that $m_2 \equiv R \pmod{m_1}$. First, suppose that $m_2 = k + 1$, that is, $x_j = z_j$, $0 \leq j \leq k$. If $m_1 = k + 1$, then $x_j = y_j$, $0 \leq j \leq k$, thus $x_0 = y_0 = z_0$, which is an absurdity. Because we are assuming that $m_1 + m_2 \leq k + g + 1$, then $m_1 = g \leq k$. From (2), it follows that $x_{k-g+1} = x_R$. Because $x_{k-g+1} = y_0$, $x_0 = z_0$, and $z_0 \neq y_0$, then $R \neq 0$. As a consequence, $x_k = x_{k-g} = x_{R-1}$, and $x_k z_0 z_1 \dots z_{R-1} = x_{R-1} x_0 x_1 \dots x_{R-1}$

is a cycle in G of length $1 \le R \le g-1$, which is a contradiction. Therefore, $g \le m_1 \le m_2 \le k$. Second, suppose that $R = 0$, or in other words, m_1 divides m_2. Then the congruence $k - m_2 + 1 \equiv k - m_1 + 1 \pmod{m_1}$ is satisfied. Now, from (2) it follows that $x_{k-m_2+1} = x_{k-m_1+1}$, yielding $z_0 = y_0$, a contradiction. Therefore, $1 \le R < m_1$.

Finally, according to (2) we have $x_{m_2} = x_R$, and from (3), we obtain $x_0 = x_{m_2} = x_R$. Furthermore, combining (2) and (3) we obtain the system of equations: $x_0 = x_{m_2} = x_R$, $x_1 = x_{m_2+1} = x_{R+1}, \dots, x_{k-m_2} = x_k = x_{k-R}, x_{k-m_2+1} = z_0 = x_{k-R+1}, \dots$. That is, the following condition is satisfied:

$$\text{if } i \equiv j \pmod{R}, \text{ then } x_i = x_j \text{ for all } 0 \le i,j \le k. \quad (4)$$

Because $m_1 + R \le m_2 \le k$ we have $0 < k+1-m_1-R < k$. Hence, from (2) it follows that $z_0 = x_{k+1-R} = x_{k+1-m_1-R}$, and from (4) $x_{k+1-m_1-R} = x_{k+1-m_1} = y_0$, that is, $y_0 = z_0$, which is a contradiction. ∎

Now, we show that the parameter ℓ is a suitable index to study how far away two different vertices of L^kG can be. To this end, it is convenient to introduce the following notation. If x and f are two different vertices of a digraph G such that $d(x,f) \le \ell$, the vertex v such that (x,v) is the first arc of the unique shortest path from x to f is denoted by $v(x \to f)$. If $d(f,x) \le \ell$, the vertex v such that (v,x) is the last arc on the unique shortest path from f to x is denoted by $v(x \leftarrow f)$. If F is a set of vertices of G and $x \notin F$ we define $v(x \to F) = \{v(x \to f) : f \in F, d(x,F) \le \ell\}$. The set $v(x \leftarrow F)$ is defined analogously. The following two lemmas are stated in [11, 18].

Lemma 3.2. *[11, 18] Let G be a loop-less digraph with $\ell = \ell(G)$ and minimum degree $\delta > 1$. Let x and f be two different vertices of G. If $d(x,f) \le \ell$, then for all $x_1 \in \Gamma^+(x)$, $x_1 \ne v(x \to f)$, it follows that $d(x_1,f) \ge \min\{d(x,f)+1, \ell\}$. Analogously, if $d(f,x) \le \ell$, for all $x'_1 \in \Gamma^-(x)$, $x'_1 \ne v(x \leftarrow f)$ it follows that $d(f,x'_1) \ge \min\{d(f,x)+1, \ell\}$.*

Lemma 3.3. *[11, 18] Let G be a loop-less digraph with $\ell = \ell(G)$ and minimum degree $\delta > 1$. Let F be a set of vertices of G such that $|F| < \delta$ and let $x \notin F$. Then for any integer $m \ge 1$, (a) there exists a path $xx_1x_2 \dots x_m$ such that for any $f \in F$, $d(x_i,f) \ge \min\{d(x_{i-1},f)+1, \ell\}$; (b) there exists a path $y_m \dots y_2y_1y$ such that for any $f \in F$, $d(f,y_i) \ge \min\{d(f,y_{i-1})+1, \ell\}$.*

The next result is similar to Lemma 3.2, but it applies to iterated line digraphs. Intuitively, it sets a lower bound on the length of a path between two given vertices other than a shortest path.

Lemma 3.4. *Let G be a digraph with girth $g \ge 2$ and parameter $\ell = \ell(G)$. Let $\mathbf{x}$ and $\mathbf{f}$ be two different vertices of L^kG, $k \ge 0$, such that $d(\mathbf{x},\mathbf{f}) \le k$. Then $d(\mathbf{x}_1,\mathbf{f}) \ge d(\mathbf{x},\mathbf{f}) + g - 1$, for all $\mathbf{x}_1 \in \Gamma^+(\mathbf{x}) \setminus \{v(\mathbf{x} \to \mathbf{f})\}$.*

Proof. Let us call d the distance from $\mathbf{x}$ to $\mathbf{f}$, that is, $d(\mathbf{x},\mathbf{f}) = d \le k$. The shortest path from vertex $\mathbf{x}$ to $\mathbf{f}$ can be represented by the sequence $x_0x_1 \dots x_ky_0y_1 \dots y_{d-1}$. Hence, vertex $\mathbf{f} = f_0f_1 \dots f_k = x_d \dots x_ky_0y_1 \dots y_{d-1}$. From Lemma 3.2 it follows that $d(\mathbf{x}_1,\mathbf{f}) \ge d+1$ because $\ell(L^kG) = k + \ell \ge k+1$. Let us consider an integer $r \ge 1$ such that $d + r = d(\mathbf{x}_1,\mathbf{f})$. The shortest path from vertex $\mathbf{x}_1$ to $\mathbf{f}$ can be represented by the sequence $x_1 \dots x_kz_0z_1z_2 \dots z_{d+r}$. First, if $d + r < k$, then $f_0 = x_{d+r+1}$, and because $f_0 = x_d$, G contains the cycle $x_d, \dots, x_{d+r+1}$ of length $r+1$, that is $r \ge g-1$. Finally, if $d+r \ge k$, then $z_{d+r-k} = f_0 = x_d$, so the cycle $x_d, \dots, x_k, z_0, \dots, z_{d+r-k}$ of length $r+1$ is contained in G. Therefore, we have proven that $d(\mathbf{x}_1,\mathbf{f}) = d + r \ge d(\mathbf{x},\mathbf{f}) + g - 1$, and the result holds. ∎

Corollary 3.5. *Let G be a digraph with girth $g \ge 2$ and parameter $\ell = \ell(G)$. Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{f}$ be three different vertices of L^kG, $k \ge 0$. Then $d(\mathbf{x}_1,\mathbf{f}) \ge \min\{g, k+1\}$, for all $\mathbf{x}_1 \in \Gamma^+(\mathbf{x}) \setminus \{v(\mathbf{x} \to \mathbf{f})\}$. Analogously, $d(\mathbf{f},\mathbf{y}_1) \ge \min\{g, k+1\}$, for all $\mathbf{y}_1 \in \Gamma^-(\mathbf{y}) \setminus \{v(\mathbf{y} \leftarrow \mathbf{f})\}$.*

Proof. As a consequence of Lemma 3.4, if $d(\mathbf{x},\mathbf{f}) \le k$, then $d(\mathbf{x}_1,\mathbf{f}) \ge d(\mathbf{x},\mathbf{f}) + g - 1 \ge g$; and if $d(\mathbf{x},\mathbf{f}) \ge k+1$, then $d(\mathbf{x}_1,\mathbf{f}) \ge \min\{d(\mathbf{x},\mathbf{f})+1, \ell+k\} \ge k+1$, because of Lemma 3.2. Then, $d(\mathbf{x}_1,\mathbf{f}) \ge \min\{g, k+1\}$. Analogously, $d(\mathbf{f},\mathbf{y}_1) \ge \min\{g, k+1\}$, so the result holds. ∎

At this point we are able to state the following result.

Lemma 3.6. *Let G be a digraph with girth $g \ge 2$ and let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{f}$ be three different vertices of L^kG, $k \ge 0$. Let $\mathbf{x}_1 \in \Gamma^+(\mathbf{x}) \setminus \{v(\mathbf{x} \to \mathbf{f})\}$ and $\mathbf{y}_1 \in \Gamma^-(\mathbf{y}) \setminus \{v(\mathbf{y} \leftarrow \mathbf{f})\}$. Then*

$$d(\mathbf{x}_1,\mathbf{f}) + d(\mathbf{f},\mathbf{y}_1) \ge \begin{cases} 2k+2 & \text{if } g \ge k+1; \\ g+k & \text{if } g \le k. \end{cases}$$

Proof. As a consequence of Corollary 3.5, $d(\mathbf{x}_1,\mathbf{f}) \ge \min\{g, k+1\}$, and $d(\mathbf{f},\mathbf{y}_1) \ge \min\{g, k+1\}$, so the result holds if $g \ge k+1$. Moreover, if $d(\mathbf{x},\mathbf{f}) \ge k+1$, then $d(\mathbf{x}_1,\mathbf{f}) \ge k+1$, where the result is also true. Furthermore, if $d(\mathbf{x},\mathbf{f}) = k$, then $d(\mathbf{x}_1,\mathbf{f}) \ge k + g - 1$ and the result also holds. In the same way it is shown that the result is valid if $d(\mathbf{f},\mathbf{y}) \ge k$. Therefore, let us assume that $1 \le d(\mathbf{x},\mathbf{f}) = d_1 \le k-1$, $1 \le d(\mathbf{f},\mathbf{y}) = d_2 \le k-1$ and $g \le k$. By Lemma 3.4, we can consider two integers $r_1, r_2 \ge g-1$ such that $d(\mathbf{x}_1,\mathbf{f}) = d_1 + r_1$ and $d(\mathbf{f},\mathbf{y}_1) = d_2 + r_2$. Also, we can suppose that $d_1 + d_2 \le k-1$; otherwise, $d_1 + r_1 + d_2 + r_2 \ge k + 2g - 2 \ge k + g$, and we are done. For $k' = k - d_1 - d_2 > 0$ we have that $L^kG = L^{d_1+d_2}L^{k'}G$, so we can represent the vertices of L^kG by sequences of $d_1 + d_2 + 1$ vertices of $L^{k'}G$. With this notation, $\mathbf{x} = x_0x_1 \dots x_{d_1+d_2}$ and $\mathbf{y} = y_0y_1 \dots y_{d_1+d_2}$. The $\mathbf{x} \to \mathbf{f} \to \mathbf{y}$ path of length $d_1 + d_2$ can be denoted by the sequence $x_0x_1 \dots x_{d_1+d_2-1}y_0y_1 \dots y_{d_1+d_2}$, that is, $y_0 = x_{d_1+d_2}$, and $\mathbf{f} = f_0f_1 \dots f_{d_1+d_2} = x_{d_1} \dots x_{d_1+d_2-1}y_0y_1 \dots y_{d_1}$. Observe that $f_{d_2-1} = x_{d_1+d_2-1}, f_{d_2} = y_0, f_{d_2+1} = y_1$. We also obtain $\mathbf{x}_1 = x_1 \dots y_0 a$, with $a \neq y_1$ because $\mathbf{x}_1 \neq v(\mathbf{x} \to \mathbf{f})$. The $\mathbf{x}_1 \to \mathbf{f}$ path of length $d_1 + r_1$ can be represented by

$$x_1 \dots y_0 a f_{d_2-r_1+1} \dots f_{d_2} \dots f_{d_1+d_2} =$$
$$x_1 \dots y_0 a f_{d_2-r_1+1} \dots x_{d_1+d_2-1} y_0 y_1 \dots y_{d_1},$$

where, if $d_2 - r_1 + 1 < 0$, the vertices f_j with $j < 0$ are those of a path from a to f_0. Also, we obtain $\mathbf{y}_1 = b y_0 y_1 \dots y_{d_1+d_2-1}$, with $b \neq x_{d_1+d_2-1}$ because $\mathbf{y}_1 \neq v(\mathbf{y} \leftarrow \mathbf{f})$. The $\mathbf{f} \to \mathbf{y}_1$ path of length $d_2 + r_2$ can be represented by

$$f_0 \dots f_{d_2} \dots f_{d_2+r_2-1} b y_0 \dots y_{d_1+d_2-1} =$$
$$f_0 \dots x_{d_1+d_2-1} y_0 y_1 \dots f_{d_2+r_2-1} b y_0 \dots y_{d_1+d_2-1},$$

where, if $d_2 + r_2 - 1 > d_1 + d_2$, then the vertices f_j with $j > d_1 + d_2$ are those of a path from $f_{d_1+d_2}$ to b. Therefore, we find in $L^{k'}G$ the two following cycles of lengths $r_1 + 1$ and $r_2 + 1$

$$y_0 a \dots x_{d_1+d_2-1} y_0 \quad \text{and} \quad y_0 y_1 \dots b y_0,$$

which satisfy the conditions of Lemma 3.1 because $a \neq y_1$. Hence, $r_1 + r_2 + 2 \geq k' + g + 2 = k - d_1 - d_2 + g + 2$ and $d(\mathbf{x}_1, \mathbf{f}) + d(\mathbf{f}, \mathbf{y}_1) = d_1 + r_1 + d_2 + r_2 \geq k + g$. ∎

As it follows from (1), if $k \geq D - 2\ell + 1$ the digraph L^kG is maximally connected, so we can guarantee that $K(s, L^kG) < \infty$, whenever $s \leq \delta - 1$. In this sense, we are ready to state the following theorem.

Theorem 3.7. *Let G be a digraph with girth $g \geq 2$, minimum degree $\delta > 1$, parameter $\ell = \ell(G)$, and diameter D. Let s be an integer such that $1 \leq s \leq \delta - 1$. If $k \geq D - 2\ell + 1$, then the s-vertex-diameter vulnerability of the iterated line digraph L^kG is $K(s, L^kG) \leq D(L^kG) + 2m$, where*

$$m = \max \times \begin{cases} \{\lceil (D - k + 1)/2 \rceil, D - k + 1 - \ell\}, & \text{if} \quad g \geq k + 1; \\ \{\lceil (D + 3 - g)/2 \rceil, D - \ell - g + 2\}, & \text{if} \quad g \leq k. \end{cases}$$

Proof. Let $F \neq \emptyset$ be a set of vertices of L^kG with $|F| = s \leq \delta - 1$ and let $\mathbf{x}_0, \mathbf{y}_0$ be two different vertices not in F. To prove the result, it is enough to find a path from $\mathbf{x}_0$ to $\mathbf{y}_0$ not containing any vertex of F and with length $D(L^kG) + 2m$. From Lemma 3.3, there exists a path $\mathbf{x}_0 \mathbf{x}_1 \dots \mathbf{x}_m$ such that, for any vertex $\mathbf{f} \in F$, and for any $i = 1, 2, \dots, m$, $d(\mathbf{x}_i, \mathbf{f}) \geq \min\{d(\mathbf{x}_{i-1}, \mathbf{f}) + 1, \ell + k\}$. In the same way there exists a path $\mathbf{y}_m \dots \mathbf{y}_1 \mathbf{y}_0$ such that, for any vertex $\mathbf{f} \in F$, and for any $i = 1, 2, \dots, m$, $d(\mathbf{f}, \mathbf{y}_i) \geq \min\{d(\mathbf{f}, \mathbf{y}_{i-1}) + 1, \ell + k\}$. That is, we have either

$$d(\mathbf{x}_m, \mathbf{f}) + d(\mathbf{f}, \mathbf{y}_m) \geq \begin{cases} d(\mathbf{x}_1, \mathbf{f}) + d(\mathbf{f}, \mathbf{y}_1) + 2m - 2, \text{ or} \\ d(\mathbf{x}_1, \mathbf{f}) + m - 1 + \ell + k, \text{ or} \\ \ell + k + d(\mathbf{f}, \mathbf{y}_1) + m - 1, \text{ or} \\ 2\ell + 2k. \end{cases}$$

Because $k \geq D - 2\ell + 1$ we have that if $d(\mathbf{x}_m, \mathbf{f}) + d(\mathbf{f}, \mathbf{y}_m) \geq 2\ell + 2k$, then $d(\mathbf{x}_m, \mathbf{f}) + d(\mathbf{f}, \mathbf{y}_m) \geq 2\ell + 2k \geq D + k + 1 = D(L^kG) + 1$. Notice that from Lemma 3.4 it follows that $d(\mathbf{x}_1, \mathbf{f}), d(\mathbf{f}, \mathbf{y}_1) \geq \min\{k + 1, g\}$. So, if $g \geq k + 1$ and $m = \max\{\lceil (D-k+1)/2 \rceil, D-k+1-\ell\}$ then $d(\mathbf{x}_m, \mathbf{f}) + d(\mathbf{f}, \mathbf{y}_m) \geq D + k + 1 = D(L^kG) + 1$. If $g \leq k$ and $m = \max\{\lceil (D + 3 - g)/2 \rceil, D - \ell - g + 2\}$, from Lemma 3.6 it follows that $d(\mathbf{x}_m, \mathbf{f}) + d(\mathbf{f}, \mathbf{y}_m) \geq g + k + 2m - 2 \geq D + k + 1 = D(L^kG) + 1$. Hence, we conclude that any path from $\mathbf{x}_m$ to $\mathbf{y}_m$ through any vertex of F has length at least $D + k + 1$. Therefore, we can assure the existence of a shortest path from $\mathbf{x}_m$ to $\mathbf{y}_m$, of length at most $D + k$, which contains no vertex of F. Hence, we have found a path from $\mathbf{x}_0$ to $\mathbf{y}_0$, $\mathbf{x}_0 \mathbf{x}_1 \dots \mathbf{x}_m \dots \mathbf{y}_m \dots \mathbf{y}_1 \mathbf{y}_0$, with length at most $D(L^kG) + 2m$ and avoiding F. ∎

To study the arc case we introduce the following notation. Let x and $e = (u, v)$ be, respectively, a vertex and an arc of a digraph G, and let us define $d(x, e) = d(x, u)$ and $d(e, x) = d(v, x)$. Now, if $1 \leq d(x, e) \leq \ell$ the first arc on the unique shortest path from x to u is denoted by $a(x \to e)$. If $x = u$ then $a(x \to e) = e$. Analogously, if $1 \leq d(e, x) \leq \ell$ the last arc on the unique shortest path from v to x is denoted by $a(x \leftarrow e)$, and $a(x \leftarrow e) = e$ if $x = v$.

Lemma 3.8. *Let G be a digraph with girth $g \geq 2$ and minimum degree $\delta > 1$. Let $\mathbf{x}$ and $\mathbf{y}$ be two different vertices, and let $\mathbf{e} = (\mathbf{u}, \mathbf{v})$ be an arc of L^kG, $k \geq 0$. Let $(\mathbf{x}, \mathbf{x}_1)$ be an arc different from $a(\mathbf{x} \to \mathbf{e})$ and $(\mathbf{y}, \mathbf{y}_1)$ be an arc different from $a(\mathbf{y} \leftarrow \mathbf{e})$. Then*

(*a*) $d(\mathbf{x}_1, \mathbf{u}), d(\mathbf{v}, \mathbf{y}_1) \geq \min\{k + 1, g - 1\}$;

(*b*) $d(\mathbf{x}_1, \mathbf{u}) + d(\mathbf{v}, \mathbf{y}_1) \geq 2k, \quad$ *if* $\quad g \geq k + 1$;
$\quad d(\mathbf{x}_1, \mathbf{u}) + d(\mathbf{v}, \mathbf{y}_1) \geq k + g - 1, \quad$ *if* $\quad g \leq k$.

Proof. (a) The result follows from Corollary 3.5 when $\mathbf{x} \neq \mathbf{u}$ and $\mathbf{y} \neq \mathbf{v}$. Furthermore, if $\mathbf{x} = \mathbf{u}$ or $\mathbf{y} = \mathbf{v}$, the result also holds because obviously $d(\mathbf{x}_1, \mathbf{u}), d(\mathbf{v}, \mathbf{y}_1) \geq g - 1$.

(b) If $k + 1 \leq g - 1$, then from case (a) we have $d(\mathbf{x}_1, \mathbf{u}) + d(\mathbf{v}, \mathbf{y}_1) \geq 2k + 2$. Moreover, if $k + 1 = g$, again from case (a), $d(\mathbf{x}_1, \mathbf{u}) + d(\mathbf{v}, \mathbf{y}_1) \geq 2g - 2 = 2k$. Therefore, assume that $g \leq k$. Notice that $\mathbf{x}_1 \neq v(\mathbf{x} \to \mathbf{u})$ and $\mathbf{y}_1 \neq v(\mathbf{v} \to \mathbf{y})$, because $(\mathbf{x}, \mathbf{x}_1) \neq a(\mathbf{x} \to \mathbf{e})$ and $(\mathbf{y}, \mathbf{y}_1) \neq a(\mathbf{y} \leftarrow \mathbf{e})$, respectively. Then the result follows directly from Lemma 3.4, if $d(\mathbf{x}, \mathbf{u}), d(\mathbf{v}, \mathbf{y}) \geq k$. Thus suppose that $d(\mathbf{x}, \mathbf{u}), d(\mathbf{v}, \mathbf{y}) \leq k - 1$; hence, $\mathbf{x}_1 \neq v(\mathbf{x} \to \mathbf{v})$ and $\mathbf{y}_1 \neq v(\mathbf{u} \to \mathbf{y})$, because between any pair of vertices of L^kG there exists at most one path of length at most $k + 1$. To finish the proof, let us distinguish the following cases:

(b1) Suppose $\mathbf{x} \neq \mathbf{u}$ and $\mathbf{y} \neq \mathbf{u}$. From Lemma 3.6, it follows that $d(\mathbf{x}_1, \mathbf{u}) + d(\mathbf{u}, \mathbf{y}_1) \geq k + g$. Therefore, $d(\mathbf{x}_1, \mathbf{u}) + d(\mathbf{v}, \mathbf{y}_1) \geq k + g - 1$, because $d(\mathbf{u}, \mathbf{y}_1) \leq 1 + d(\mathbf{v}, \mathbf{y}_1)$.

(b2) Suppose $\mathbf{x} = \mathbf{u}$ and $\mathbf{y} \neq \mathbf{v}$. Then $\mathbf{x} \neq \mathbf{v}$ and from Lemma 3.6, it follows that $d(\mathbf{x}_1, \mathbf{v}) + d(\mathbf{v}, \mathbf{y}_1) \geq k + g$. Therefore, $d(\mathbf{x}_1, \mathbf{u}) + d(\mathbf{v}, \mathbf{y}_1) \geq k + g - 1$, because $d(\mathbf{x}_1, \mathbf{v}) \leq d(\mathbf{x}_1, \mathbf{u}) + 1$.

(b3) Suppose $\mathbf{x} = \mathbf{u}$ and $\mathbf{y} = \mathbf{v}$. Consider the cycle $C_1 = \mathbf{x}\mathbf{x}_1 \to \mathbf{x}$, where $\mathbf{x}_1 \to \mathbf{x}$ is a shortest path, and the cycle $C_2 = \mathbf{y} \to \mathbf{y}_1\mathbf{y}$, where $\mathbf{y} \to \mathbf{y}_1$ is a shortest path. Thus, C_1 and C_2 are two cycles of L^kG joined by the arc $\mathbf{e} = (\mathbf{x}, \mathbf{y})$. As $k \geq g \geq 2$, the vertices of L^kG can be represented as sequences of two vertices of $L^{k-1}G$. With this notation $\mathbf{x} = x_0x_1$, $\mathbf{x}_1 = x_1a$ and $\mathbf{y} = x_1b$, where $x_0, x_1, a, b \in L^{k-1}G$ and $a \neq b$, because $\mathbf{x}_1 \neq v(\mathbf{x} \to \mathbf{y})$. Then $L^{k-1}G$ must contain two cycles $C_1' = x_1a \dots x_1$ and $C_2' = x_1b \dots x_1$ whose lengths are $1 + d(\mathbf{x}_1, \mathbf{x})$ and $d(\mathbf{y}, \mathbf{y}_1) + 1$, respectively. From Lemma 3.1, the sum of their lengths is at least $(k - 1) + g + 2$, so $d(\mathbf{x}_1, \mathbf{x}) + d(\mathbf{y}, \mathbf{y}_1) \geq k + g - 1$. ∎

As a direct consequence of (1) it follows that the digraph L^kG is maximally arc connected when $k \geq D - 2\ell$ so, as in the vertex case, we can assure that $\Lambda(s, L^kG) < \infty$ whenever $s \leq \delta - 1$. The following theorem is the arc version of Theorem 3.7.

Theorem 3.9. *Let G be a digraph with girth $g \geq 2$, minimum degree $\delta > 1$, parameter $\ell = \ell(G)$, and diameter D. Let s be an integer such that $1 \leq s \leq \delta - 1$. Then, if $k \geq D - 2\ell$, the arc-diameter vulnerability of the iterated line digraph L^kG is $\Lambda(s, L^kG) \leq D(L^kG) + 2m$ where*

$$m = \max \times \begin{cases} \{\lceil (D - k + 2)/2 \rceil, D - k - \ell + 1\}, & \text{if } g \geq k + 1; \\ \{\lceil (D + 3 - g)/2 \rceil, D - \ell - g + 2\}, & \text{if } g \leq k. \end{cases}$$

Proof. Let F be a set of arcs of L^kG with $|F| = s$, $1 \leq s \leq \delta - 1$. Consider the sets of vertices $F_1 = \{\mathbf{u} : (\mathbf{u}, \mathbf{v}) \in F\}$, $F_2 = \{\mathbf{v} : (\mathbf{u}, \mathbf{v}) \in F\}$ and two different vertices $\mathbf{x}, \mathbf{y}$ of L^kG. We will find a path from $\mathbf{x}$ to $\mathbf{y}$ not containing any arc of F and with length at most $D(L^kG) + 2m$.

Let us define the sets $a(\mathbf{x} \to F) = \{a(\mathbf{x} \to \mathbf{e}) : \mathbf{e} \in F, d(\mathbf{x}, \mathbf{e}) \leq \ell + k\}$ and $a(\mathbf{y} \leftarrow F) = \{a(\mathbf{y} \leftarrow \mathbf{e}) : \mathbf{e} \in F, d(\mathbf{e}, \mathbf{y}) \leq \ell + k\}$. Because $s \leq \delta - 1$, there exist arcs $(\mathbf{x}, \mathbf{x}_1) \notin a(\mathbf{x} \to F)$ and $(\mathbf{y}_1, \mathbf{y}) \notin a(\mathbf{y} \leftarrow F)$. Moreover, if $\mathbf{x} \notin F_1$, then $\mathbf{x}_1 \notin F_1$; and, if $\mathbf{x} \in F_1$, then we can take a vertex $\mathbf{x}_1 \notin F_1$ because $|a(\mathbf{x} \to F)| \leq s < \delta$. In the same way, we show that we can take $\mathbf{y}_1 \notin F_2$. Now, from Lemma 3.3 there exists a path $\mathbf{x}\mathbf{x}_1 \dots \mathbf{x}_m$, such that, for any $\mathbf{u} \in F_1$ and for any $i = 2, \dots, m$, $d(\mathbf{x}_i, \mathbf{u}) \geq \min\{d(\mathbf{x}_1, \mathbf{u}) + i - 1, \ell + k\}$. Analogously, there exists a path $\mathbf{y}_m \dots \mathbf{y}_1\mathbf{y}$, such that, for any $\mathbf{v} \in F_2$ and for any $i = 2, \dots, m$, $d(\mathbf{v}, \mathbf{y}_i) \geq \min\{d(\mathbf{v}, \mathbf{y}_1) + i - 1, \ell + k\}$. That is, we have either

$$d(\mathbf{x}_m, \mathbf{u}) + 1 + d(\mathbf{v}, \mathbf{y}_m) \geq \begin{cases} d(\mathbf{x}_1, \mathbf{u}) + d(\mathbf{v}, \mathbf{y}_1) + 2m - 1, \text{ or} \\ d(\mathbf{x}_1, \mathbf{u}) + m + \ell + k, \text{ or} \\ \ell + k + d(\mathbf{v}, \mathbf{y}_1) + m, \text{ or} \\ 2\ell + 2k + 1 \end{cases}$$

First, notice that if $d(\mathbf{x}_m, \mathbf{u}) + 1 + d(\mathbf{v}, \mathbf{y}_m) \geq 2\ell + 2k + 1$, then $d(\mathbf{x}_m, \mathbf{u}) + 1 + d(\mathbf{v}, \mathbf{y}_m) \geq D(L^kG) + 1$, because $k \geq D - 2\ell$. Moreover, because $(\mathbf{x}, \mathbf{x}_1) \neq a(\mathbf{x} \to \mathbf{e})$ and $(\mathbf{y}_1, \mathbf{y}) \neq a(\mathbf{y} \leftarrow \mathbf{e})$ (recall $\mathbf{e} = (\mathbf{u}, \mathbf{v})$) we can apply Lemma 3.8 and, hence, $d(\mathbf{x}_1, \mathbf{u}), d(\mathbf{v}, \mathbf{y}_1) \geq \min\{k + 1, g - 1\}$. So, if $g \geq k + 1$ and $m = \max\{\lceil (D - k + 2)/2 \rceil, D - k - \ell + 1\}$, then $d(\mathbf{x}_m, \mathbf{u}) + 1 + d(\mathbf{v}, \mathbf{y}_m) \geq D + k + 1 = D(L^kG) + 1$. If $g \leq k$, then from Lemma 3.8 it follows that $d(\mathbf{x}_1, \mathbf{u}) + d(\mathbf{v}, \mathbf{y}_1) \geq g + k - 1$. Therefore, if $m = \max\{\lceil (D + 3 - g)/2 \rceil, D + 2 - \ell - g\}$, $d(\mathbf{x}_m, \mathbf{u}) + 1 + d(\mathbf{v}, \mathbf{y}_m) \geq D + k + 1 = D(L^kG) + 1$. Thus, we can now assure the existence of a shortest path from $\mathbf{x}_m$ to $\mathbf{y}_m$ of length at most $D + k$, which contains no arc of F. Hence, G must contain a path from $\mathbf{x}$ to $\mathbf{y}$ (the path $\mathbf{x}\mathbf{x}_1 \dots \mathbf{x}_m \dots \mathbf{y}_m \dots \mathbf{y}_1\mathbf{y}$) with length at most $D(L^kG) + 2m$ avoiding F. ∎

4. APPLICATIONS

In this section we apply the above results to some important families of iterated line digraphs. Diameter vulnerability of each of these families has been studied in the past by using particular methods for each one. The advantage of our results is that they can be applied to each of these families just taking into account the girth and parameter ℓ of the original digraph. Furthermore, if the number of iterations k is not small, then our results improve the previous bounds given until now.

4.1. Kautz Digraphs

The Kautz digraph $K(d, D)$ is the iterated line digraph $L^{D-1}K_{d+1}$, where K_{d+1} denotes the complete symmetric digraph on $d + 1$ vertices [17]. Diameter vulnerability of the Kautz digraphs has been studied by finding disjoint paths between any pair of vertices [7, 16].

Now, we apply Theorem 3.7 and Theorem 3.9 to this family.

Corollary 4.1. *Let $G = K(d, D)$ be the Kautz digraph of degree $d \geq 2$ and diameter $D \geq 2$. Let s be an integer such that $1 \leq s \leq d - 1$. Then the diameter vulnerability of G is*

$$K(s, G) \leq D(G) + 2;$$
$$\Lambda(s, G) \leq D(G) + 2.$$

The bounds obtained in the above corollary coincide with those presented in [10].

4.2. Dense Bipartite Digraph

For any positive integers d, n, with $d \leq n$, the *dense bipartite digraph* $BD(d, n)$ introduced in [12] has the set of vertices $V = \mathbb{Z}_2 \times \mathbb{Z}_n = \{(\alpha, i); \alpha \in \mathbb{Z}_2, i \in \mathbb{Z}_n\}$ where $\mathbb{Z}_n$ denotes the integers modulo n and each vertex (α, i) is adjacent to the vertices of $\Gamma^+(\alpha, i) = \{(1 - \alpha, (-1)^\alpha d(i + \alpha) + t);\ t = 0, 1, \dots, d - 1\}$. The digraphs $BD(d, d^{k-1} + d^{k-3})$ can also be obtained as iterated line digraphs of $BD(d, d^2 + 1)$, which has diameter $D = 3$, $g = 4$, and parameter $\ell = 2$.

Corollary 4.2. *Let $d \geq 2$, $k \geq 3$ and s, $1 \leq s \leq d-1$, be three integers. Then the diameter vulnerability of the digraph $G = BD(d, d^{k-1} + d^{k-3})$ is*

$$\begin{array}{lll} K(s,G) \leq 7, & \Lambda(s,G) \leq 8, & \text{if } 3 \leq k \leq 6; \\ K(s,G) \leq k+2, & \Lambda(s,G) \leq k+2, & \text{if } k \geq 7. \end{array}$$

Proof. Consider the digraph $BD(d, d^2+1)$, which has a diameter of 3, a girth of 4, and a parameter ℓ equal to 2. The result follows by applying Theorem 3.7 and Theorem 3.9 to the digraph $BD(d, d^2+1)$, taking into account that $G = L^{k-3}BD(d, d^2+1)$ has a diameter k. ∎

Diameter vulnerability of this family was also studied in [19], where it was proven that the diameter increases by at most one unit if fewer than $d-2$ vertices are removed, and by at most two when $d-2$ or $d-1$ vertices are deleted. For most cases our results improve the known bounds, while for the diameter equal to 3 or $k = 4$, the known bounds are exceeded by at most two units. Moreover, if $k \geq 7$, the bounds presented here improve by one unit those obtained from the bounds presented in [10].

4.3. The deBruijn Generalized Cycle

The deBruijn generalized cycle $BGC(p, d, d^{k+1})$ is defined as the k iterated line digraph of $C_p \otimes K_d^+$ where C_p denotes the directed cycle of length p, K_d^+ is the complete digraph on d vertices with a loop on each vertex. The conjunction of the cycle C_p with an arbitrary digraph H, denoted by $C_p \otimes H$, has the set of vertices $\mathbb{Z}_p \times V(H)$ where a vertex (α, x) is adjacent to the vertices $(\alpha+1, y)$ for any y adjacent from x in the digraph H. Observe that $C_p \otimes H$ is a generalized p-cycle for any digraph H considered in [15] and in [2], its connectivity being studied in the latter reference. The digraph $BGC(p, d, d) = C_p \otimes K_d^+$, the complete p-generalized cycle of degree d, has a diameter and girth equal to p and a parameter $\ell = 1$. The family of deBruijn generalized cycles contains the deBruijn digraphs [5], the Reddy-Pradhan-Kuhl digraphs [20], and the butterflies [1], all of them being very important interconnection network models.

Corollary 4.3. *Let $G = BGC(p, d, d^{k+1})$ be the deBruijn generalized cycle with $d \geq 2$, $p \geq 2$ and $k \geq p-2$. Then its diameter vulnerability for $1 \leq s \leq d-1$ is*

$$\begin{array}{ll} K(s,G) \leq D(G)+2, & \text{if } k = p-1; \\ K(s,G) \leq D(G)+3, & \text{if } k \geq p; \\ \Lambda(s,G) \leq D(G)+p+2-k, & \text{if } p-2 \leq k \leq p-1; \\ \Lambda(s,G) \leq D(G)+3, & \text{if } k \geq p. \end{array}$$

Diameter vulnerability of $BGC(p, d, d^{k+1})$ was previously studied in [9] by finding disjoint paths between any pair of different vertices. If $k = p-1$, the bounds obtained from the above corollary coincide with the results in [9]; for all other values, the new bounds differ from those found in [9] by one unit.

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