Fault-Tolerant Routings in Large Generalized Cycles

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Abstract

A generalized p-cycle is a digraph whose set of vertices is partitioned in p parts that can be ordered in such a way that a vertex is adjacent only to vertices in the next part. The families of $BGC(p,d,d^k)$ *and* $KGC(p,d,d^{p+k}+d^k)$ *are the largest known p-cyles for their degree and diameter.*

In this paper we present routing algorithms for both families. Such algorithms route over paths of length at most the value of the diameter plus two units. Moreover, this bound is attained only in the case that the number of faulty elements (nodes or links) is maximum.

1. Introduction

Interconnection networks have been usually modeled by graphs and digraphs. The switching elements or processors are represented by the vertices. The communication links are represented by edges (if they are bidirectional) or arcs (if they are unidirectional). We are only concerned with directed graphs, called digraphs for short.

In the design of such networks several requirements should be taken into account. Some optimization problems on graphs and digraphs arise when translating these requirements into the above model. One of them is to find digraphs with large order and fixed degree d and diameter D((d,D)-digraph problem).

For the (d,D)-digraph problem the best known general solutions are the de Bruijn and Kautz families of digraphs. In the case of bipartite digraphs, the family of $BD(d,d^{D-3}+d^{D-1})$ proposed by Fiol and Yebra in [4], is the best solution known so far. Lately, this problem has been studied in the case of generalized cycles. That is, for digraphs whose set of vertices is partitioned in several parts that can be ordered in such a way that a vertex is adjacent only to vertices in the next part. In this case, the best known solutions are families of generalized cycles proposed by Gomez, Padró and Perennes in [6]. In fact, the $BGC(p,d,d^k)$ and $KGC(p,d,d^{p+k}+d^k)$ are the largest p-cycles for their degree and diameter. These families are a generalization of the proposed solutions for the general and bipartite cases.

Other requirement for an interconnection network is that the system still works if some elements (nodes or links) fail. This mean that the network must be fault-tolerant. Furthermore, it is desirable that the network still communicate with reasonable efficiency. In order to achive this requirement is necessary to find routing algorithms that could communicate when same elements are faulty, but the network still connects every pair of nodes.

Fault tolerant routings in de Bruijn and Kautz digraphs has been studied in [9]. For the bipartite digraphs $BD(d,d^{D-3}+d^{D-1})$ routing was studied in [10] with similar techniques.

In this paper we study routing algorithms for large generalized cycles. In the next section we present the most relevant notation and definitions to use in the following. In Section 3 and Section 4 we construct the set of paths that the algorithms will use to route. In Section 5 and Section 6 we give a complete description of the algorithms in $BGC(p,d,d^k)$ and $KGC(p,d,d^{p+k}+d^k)$ respectively.

2. Definitions, notation and known results

We are concerned only in *digraphs*. See [2] for the definitions of the concepts about digraphs that are not defined here.

In the *line digraph* LG of a digraph G each vertex

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represents an arc of G, that is, $V(LG) = \{uv/(u,v) \in A(G)\}$. A vertex uv is adjacent to a vertex vw if $v = w$, that is, whenever the arc (u,v) of G is adjacent to the arc (w,z). The maximum and minimum out and in-degrees of LG are equal to those of G. Therefore, if G is a strongly connected digraph different from a directed cycle, then the diameter of LG is the diameter of G plus one.

The *k-iterated line digraph*, $L^k(G)$, is defined recursively by $L^k(G) = LL^{k-1}(G)$, beginning with $L^1(G) = LG$. Note that a vertex x of $L^k(G)$ may be represented as a sequence $x_0, x_1, \ldots, x_k$ of vertices of G such that $(x_i, x_{i+1}) \in A, 0 \le i \le k-1$. Then an arc in $L^k(G)$ can be represented by a sequence of $k+1$ vertices of G. In general, a path of length h can be represented as a sequence of $k+h$ vertices of G.

A *generalized p-cycle* is a digraph G such that its set of vertices can be partitioned in p parts, $V(G) = \cup_{\alpha \in \mathbf{Z}_p} V_\alpha$, in such a way that the vertices in the partite set V_α are only adjacent to vertices in $V_{\alpha+1}$, where the sum is in $\mathbf{Z}_p$. If G is strongly connected, $\Gamma^+(V_\alpha) = V_{\alpha+1}$. Bipartite digraphs are generalized p-cycles with $p = 2$.

The *conjonction* of a directed cycle of length p with a digraph $G = (V, A)$, $C_p \otimes G$, has set of vertices $\mathbf{Z}_p \times V$ and a vertex (α, x) is adjacent to the vertices $(\alpha+1, y)$ for any y adjacent from x in G. Observe that $C_p \otimes G$ is a generalized p-cycle for any digraph G. The diameter of $C_p \otimes G$ is $h + p - 1$, where h is the minimum integer such that for any pair of vertices x, y,not necessarily different, of G, there is a path from x to y with length $l \le h$ and $l \equiv h(modp)$. The line digraph of $C_p \otimes G$ is isomorphic to $C_p \otimes LG$. Let $G = (V, A)$ be a digraph and let ϕ an automorphism of G. The generalized cycle $C_p \otimes_\phi G$ is defined as follows: its set of vertices is $\mathbf{Z}_p \times V$ and the adjacency rule is given by: $\Gamma^+(V_{\alpha,x}) = (\alpha + 1, \Gamma^+_\phi(x)) = \{(\alpha+1, y) : (\phi_\alpha(x), y) \in A\}$.

In [6] Gómez, Padró and Perennes studied the problem of finding regular generalized p-cycles with given degree and diameter that have large order. They gave a bound for this problem and some families of generalized cycles wich are very close to this bound. These families are the *de Bruijn generalized cycles*, $BGC(p, d, d^{k+1})$ wich are defined by $C_p \otimes GB(d, d^{k+1})$, and the *Kautz generalized cycles*, $KGC(p, d, d^{p+k} + d^k)$ defined by $C_p \otimes_\phi GB(d, d^{p+k} + d^k)$, where ϕ is an automorphism in $\mathbf{Z}_p$ definded by $\phi(x) = -x - 1$. These are the largest known p-cycles for their degree and diameter. We recall here some interesting properties of these families. The generalized cycle $BGC(p, d, d^{k+1}) = C_p \otimes B(d, k+1) = C_p \otimes L^k K_d$ is d-regular, has d^{k+1} vetices in each partite set, and diameter $p + k$. The p-cycles $KGC(p, d, d^{p+k} + d^k) = L^k KGC(p, d, p(d^p + 1))$ are d-regular, have $d^{p+k} + d^k$ vertices in each partite set, and diameter $2p + k - 1$.

These families contain these wich were proposed as solutions to the problems mentioned above restricted to the cases $p = 1$ (general digraphs) and $p = 2$ (bipartite digraphs). Actually, $BGC(1, d, n)$ and $KGC(1, d, n)$ coincide with $GB(d, n)$ and $GK(d, n)$ respectively. Also, $B(d, D) = BGC(1, d, d^D)$ and $K(d, D) = KGC(1, d, d^D + d^{D-1})$. The bipartite digraphs $BD(d, n)$, introduced by Fiol and Yebra in [4], are the same generalized cycles $KGC(2, d, n)$. The bipartite digraphs $BD(d, d^{D-1} + d^{D-3}) = KGC(2, d, d^{D-p+1} + d^{D-2p+1})$ have large order for their degree and diameter. Finally, observe that $BGC(p, d, d^p)$ wich has diameter $2p - 1$ and order pd^p, is isomorphic to the *directed butterfly*, $B_d(p)$ [1]. Therefore, the directed butterfly is an iterated line digraph, $B_d(p) = C_p \otimes L^{p-1} K^*_d$.

3. Paths in $BGC(p, d, d^{k+1})$

We are going to construct d independent paths between any pair of vertices of $BGC(p, d, d^{k+1})$, of length at most $p + k + 2$. Moreover, at most one of them has length $p + k + 2$.

Let u, v two adjacent vertices of $BGC(p, d, d^{k+1}) = L^k(C_p \otimes K^+_d)$:

$$u = (x_0, x_1, \ldots, x_k)$$
$$v = (x_1, \ldots, x_k, x_{k+1})$$

with $x_i \in C_p \otimes K^+_d, \forall i = 0, \ldots, k+1$.
An evident path from u to v is the arc (u,v): $[x_0, x_1, \ldots, x_k, x_{k+1}]$. So, we must find $d-1$ paths from u to v, not containing the arc (u, v) and with general expression:

$$[x_0, x_1, \ldots, x_k, a_1, a_2, \ldots, a_r, x_1, x_2, \ldots, x_{k+1}]$$

with $r \le p + 1$.
That is, we need $d-1$ paths from x_k to x_1 in $C_p \otimes K^+_d$, with length at most $p+2$ (and at most one with length $p+2$). These paths must be independent with the first one, so we ask: $a_1 \ne x_{k+1}$ and $a_r \ne x_0$.
As $[x_1, \ldots, x_k]$ is a path in $C_p \otimes K^+_d$, if $x_1 \in V_i$ then $x_k \in V_j$ with $j =_p k + i - 1$. To consider this, we study different cases, depending on the value of k in $\mathbf{Z}_p$:

- 1. $k \equiv_p 0$: $d(x_k, x_1) = 1$ and we can assume that $x_k \in V_0$ and $x_1 \in V_1$.

  If $x_0 \ne x_k$ and $x_1 \ne x_{k+1}$: One path is the arc (x_k, x_1). The other $d-2$ paths have pattern: $[x_k, \alpha^s_1, \alpha^s_2, \ldots, \alpha^s_{p-1}, \alpha^s_p, x_1]$, with $\alpha^s_1 \ne x_1, x_{k+1}$, $\alpha^s_p \ne x_k, x_0$ and $1 \le s \le d-2$.

If $x_0 = x_k$ and $x_1 = x_{k+1}$: All the paths are taken with pattern $[x_k, \alpha_1^s, \alpha_2^s, \ldots, \alpha_{p-1}^s, \alpha_p^s, x_1]$, with $\alpha_1^s \neq x_1$, $\alpha_p^s \neq x_0$ and $1 \leq s \leq d-1$.

If $x_0 \neq x_k$ and $x_1 = x_{k+1}$: We take a path $[x_k, \alpha_1^1, \alpha_2^1, \ldots, \alpha_{p-1}^1, x_k, x_{k+1}]$ with $\alpha_1^1 \neq x_{k+1}$, and $d-2$ paths $[x_k, \alpha_1^s, \alpha_2^s, \ldots, \alpha_{p-1}^s, \alpha_p^s, x_1]$, with $\alpha_1^s \neq x_1$, $\alpha_p^s \neq x_k, x_0$ and $2 \leq s \leq d-1$.

If $x_0 = x_k$ and $x_1 \neq x_{k+1}$: One path with form $[x_k, x_1, \alpha_2^1, \ldots, \alpha_p^1, x_1]$ with $\alpha_p^s \neq x_0$. The remaining paths are $[x_k, \alpha_1^s, \alpha_2^s, \ldots, \alpha_{p-1}^s, \alpha_p^s, x_1]$, with $\alpha_p^s \neq x_k$, $\alpha_1^s \neq x_1, x_{k+1}$ and $2 \leq s \leq d-1$.

- 2. $k \equiv_p r$, $1 \leq r \leq p-1$, $p \geq 4$: $d(x_k, x_1) = p - r + 1$ and we can assume that $x_k \in V_r$, $x_1 \in V_1$. All paths have pattern $[x_k, \alpha_{r+1}^s, \alpha_{r+2}^s, \ldots, \alpha_p^s, x_1]$, with $\alpha_1^s \neq x_{k+1}$, $\alpha_p^s \neq x_0$, and $1 \leq s \leq d-1$.

- 3. $k \equiv_p p-1$: $d(x_k, x_1) = 2$ so we assume that $x_k \in V_{p-1}$ and $x_1 \in V_1$.

  If $x_{k+1} \neq x_0$: We take $d-2$ paths with pattern $[x_k, \alpha_1^s, x_1]$, with $\alpha_1^s \neq x_0, x_{k+1}$ and $2 \leq s \leq d-1$, and a path $[x_k, x_0, \alpha_1^1, \alpha_2^1, \ldots, \alpha_{p-1}^1, x_{k+1}, x_1]$ with $\alpha_1^1 \neq x_1$, $\alpha_{p-1}^1 \neq x_k$.

  If $x_{k+1} = x_0$: All paths with the pattern $[x_k, \alpha_0^s, x_1]$, with $\alpha_0^s \neq x_0$, $1 \leq s \leq d-1$.

Proposition 3.1
If $Q_s = [x_0, x_1, \ldots, x_k, \alpha_s^1, \ldots, \alpha_s^r, x_1, \ldots, x_k, x_{k+1}]$ *and* $Q_t = [x_0, x_1, \ldots, x_k, \alpha_t^1, \ldots, \alpha_t^r, x_1, \ldots, x_k, x_{k+1}]$. *If* $\{\alpha_s^1, \ldots, \alpha_s^r\} \cap \{\alpha_t^1, \ldots, \alpha_t^r\} = \phi$ *and* $1 \leq r \leq p$, *then* Q_s *and* Q_t *are disjoint.*

Proposition 3.2
If Q_s *has the same pattern that in Prop. 3.1 and* Q_t *is* $[x_0, x_1, \ldots, x_k, \alpha_t^1, \ldots, \alpha_t^r, \ldots, \alpha_t^{p+r}, x_1, \ldots, x_k, x_{k+1}]$. *If* $\{\alpha_s^1, \ldots, \alpha_s^r\} \cap \{\alpha_t^1, \ldots, \alpha_t^{p+r}\} = \phi$ *and* $1 \leq r \leq p$, *then* Q_s *and* Q_t *are disjoint.*

To prove these propositions we state:

Lemma 3.3
Let u, v *two vertices in* $V(BGC(p, d, d^{k+1}))$ *with expressions in terms of* $C_p \otimes K_d^+$:

$$u = b_1 \ldots b_n a_1 \ldots a_j c_1 \ldots c_{k+1-n-j}$$
$$v = a_1' \ldots a_{j'}' c_1' \ldots c_{k+1-n-j'}' b_1 \ldots b_n$$

If $\{a_1 \ldots a_j\} \cap \{c_1' \ldots c_{k+1-n-j'}'\} = \phi$ *and* $j' < j$ *or* $\{a_1' \ldots a_{j'}'\} \cap \{c_1 \ldots c_{k+1-n-j}\} = \phi$ *and* $n < k+1-n-j$ *then* $u \neq v$.
Moreover, if the expressions for u, v *can be ordered in the above pattern,* $u \neq v$.

Proof: If $u = v$, equaling term by term the expressions for u and v we can construct the *equivalence digraph* [9]. There, $b_1, \ldots, b_n$ have in-degree 1 and out-degree 1, $a_1, \ldots, a_j, c_1, \ldots, c_{k+1-n-j}$ in-degree 0 and out-degree 1 and $a_1', \ldots, a_{j'}', c_1', \ldots, c_{k+1-n-j'}'$ in-degree 1 and out-degree 0. If $j' < j$ or $n < k+1-n-j$, there exist α, β such that $1 \leq \alpha \leq k+1-n-j$, $1 \leq \beta \leq j'$ and $a_\alpha' = c_\beta'$ wich is not possible.
Besides, if the expressions for u, v achive the pattern given, the situation in the equivalence digraph will be the same.

Proof of Proposition 3.1:
Let $q_{s,j}$ the j-th vertex after u in Q_s:

$$\begin{cases} x_j, \ldots, x_k, \alpha_s^1, \ldots, \alpha_s^j, j = 1 \ldots r \\ x_j, \ldots, x_k, \alpha_s^1, \ldots, \alpha_s^r, x_1 \ldots, x_{j-r}, j = r+1 \ldots k \\ \alpha_s^{j-k}, \ldots, \alpha_s^r, x_1 \ldots, x_{j-r}, j = k+1 \ldots k+r \end{cases}$$

and $q_{t,i}$ the i-th after u in Q_t:

$$\begin{cases} x_i, \ldots, x_k, \alpha_t^1, \ldots, \alpha_t^i, i = 1 \ldots r \\ x_i, \ldots, x_k, \alpha_t^1, \ldots, \alpha_t^r, x_1 \ldots, x_{i-r}, i = r+1 \ldots k \\ \alpha_t^{i-k}, \ldots, \alpha_t^r, x_1 \ldots, x_{i-r}, i = k+1 \ldots k+r \end{cases}$$

We have to show that $q_{s,j} \neq q_{t,i}$. By the simmetry of the paths and because we are on a p-cycle, it suffices to consider the case $j \leq i$ and $i \equiv_p j$. With these considerations, it is enough to apply the Lemma 3.3 to conclude that Q_s and Q_t are disjoint.

Proof of Proposition 3.2:
If $r = 0$ is trivial. If $r > 0$, from Prop. 3.1, $Q_s = [x_0, x_1, \ldots, x_k, \alpha_s^1, \ldots, \alpha_s^r, x_1, \ldots, x_k, x_{k+1}]$ and $Q_t = [x_0, x_1, \ldots, x_k, \alpha_t^1, \ldots, \alpha_t^r, x_1, \ldots, x_k, x_{k+1}]$ are disjoint. Then Q_s does not have any vertex in common with the first $r+1$ of Q_t. Analogously, with $[x_0, x_1, \ldots, x_k, \alpha_t^{r+1}, \ldots, \alpha_t^{r+p}, x_1, \ldots, x_k, x_{k+1}]$ instead of Q_t, we have that Q_s does not have any common vertex with the last $p+1$ of Q_t.
Now, we have to prove that Q_s does not have any common vertex with the ones of Q_t in positions from $r+2$ to $k+p+1$. As we do in the proof of Prop. 3.1, we compare the expression for $q_{s,j}$:

$$\begin{cases} x_{j-1} \ldots \alpha_1^s \ldots \alpha_{j-1}^s, j = 1 \ldots p+1 \\ x_{j-1} \ldots \alpha_1^s \ldots x_{j-p-2}, j = p+2 \ldots k \\ \alpha_{j-k-1}^s \ldots \alpha_{p+1}^s \ldots x_{j-p-2}, j = k+1 \ldots p+k+1 \end{cases}$$

with the expression for $q_{t,i}$:

$$\begin{cases} x_{i-1} \ldots \alpha_1^t \ldots \alpha_{i-1}^t, i = r+2 \ldots, p+r+1 \\ x_{i-1} \ldots \alpha_1^t \ldots x_{i-1-p-r}, i = p+r+2 \ldots k+1 \\ \alpha_{i-k-1}^t \ldots \alpha_{p+r}^t \ldots x_{i-1-p-r}, i = k+2 \ldots p+k+1 \end{cases}$$

By the condition $\{\alpha_s^1, \ldots, \alpha_s^r\} \cap \{\alpha_t^1, \ldots, \alpha_t^{r+p}\} = \phi$ vertices with $j = 1 \ldots p+1$ and $i = r+2 \ldots, p+r+1$

($j = k+1 \ldots p+k+1$ and $i = k+2 \ldots p+k+1$ or with $j = p+2 \ldots k$ and $i = p+r+2 \ldots k+1$) cannot coincide. For other cases, it suffices to apply the Lemma 3.3.

Theorem 3.4
Let $u, v \in V(BGC(p, d, k+1))$. There exist d disjoint paths from u to v with length less or equal than $p+k+2$, and at most one with length $p+k+2$.

If u, v are adjacent the theorem is valid by the above propositions. If not, is easy prove it by induction on k.

4. Paths in $KGC(p, d, d^{p+k} + d^k)$

In the first part of this section, we deal with disjoint paths in $KGC(p, d, d^p + 1)$. In the second, from these paths we prove the existence of the paths needed in $KGC(p, d, d^{p+k} + d^k)$.

Let us see how to construct the paths needed in $KGC(p, d, d^p + 1)$.

We are going to construct $d-1$ disjoint paths of length at most $2p+1$ between every pair of vertices x, y in $KGC(p, d, d^p + 1)$. Moreover, these paths avoid two given arcs, one from x and the other to y.

Proposition 4.1
Let $x, y \in V(KGC(p, d, d^p + 1))$, different or not. There exist d disjoint paths from x to y with length at most $2p$.

Proof. Suppose $x \in V_0$ and $y \in V_k$, $1 \leq k \leq p$ and prove by induction on k.

If $k = 1$ and (x, y) is an arc, $\Gamma^+(x) = \{y, z_2, \ldots, z_d\}$. As there is a unique path of length p from any z_i to y, we have a path of length 1 and $d-1$ paths with length $p+1 \leq 2p$. Analogously, if (x, y) is not an arc, $\Gamma^+(x) = \{z_1, z_2, \ldots, z_d\}$. As there is a unique path of length p from any z_i to y, we have d paths of length $p+1 \leq 2p$.

Now suppose that for any $y \in V_l$, $1 \leq l \leq k-1$, there exist d disjoint paths from x to y, with length at most $p+l$.
Let $y \in V_k$ and $\Gamma^-(y) = \{v_1, v_2, \ldots, v_d\}$. As $\Gamma^-(y)$ is in V_{k-1}, there exist d disjoint paths from x to each vertex v_i, with length $p+k-1$. From these, we construct a set of d disjoint paths. First, we take an arbitrary path from x to v_1, wich begin by the arc (x, z_1). Then, we add a path from x to v_2 wich not contain the vertex z_1. (These paths exist because there are d disjoint paths from x to v_2). Suppose that this path begin with the arc (x, z_2). Iterating this procedure d times we obtain d paths from x to each vertex in $\Gamma^-(y)$, with length $p+k-1$.

Clearly, the paths constructed by adding the arc (v_i, y) to the path from x to v_i are disjoint.

Proposition 4.2
Let $x, y \in V(KGC(p, d, d^p+1))$ in non-adjacent partite sets, $x' \in V(\Gamma^+(x))$ and $y' \in \Gamma^-(y)$. There exist $d-1$ disjoint paths of length at most $2p$ from x to y, avoiding the arcs (x, x') and (y', y).

Proof. By the Prop. 4.1, we know d disjoint paths of length at most $2p$ from x to y. Also, as the out-degree of x and the in-degree of y are d, these paths contain the arcs (x, x') and (y', y), so:

If the arcs (x, x') and (y', y) belong to the same path, we discard it, and still have $d-1$ disjoint paths from x to y with length at most $2p$.

If the arcs (x, x') and (y', y) are in different paths, we discard both, and define another path. Suppose $x \in V_0$ and $y \in V_k$, with $k \neq 1, p-1$. If (x, x') is in the path of minimum length (k) from x to y, we have: $[x, x', \alpha_2^s, \ldots, \alpha_{k-1}^s, y]$ $(\alpha_{k-1}^s \neq y')$ and $[x, \alpha_1^t, \ldots, \alpha_p^t, \ldots, \alpha_{p+k-2}^t, y', y]$ $(\alpha_1^t \neq x')$. From these, we construct a new one, replacing the arc (x, x') with a path of length $p+1$ and using the vertex α_1^t: $[x, \alpha_1^t, \ldots, x', \alpha_2^s, \ldots, \alpha_{k-1}^s, y]$. This path is disjoint with the other $d-2$ because $[x', \alpha_2^s, \ldots, \alpha_{k-1}^s, y]$ does not intersect any path, and $[\alpha_1^t, \ldots, x']$ have length p and α_1^t is not used from x by any other path. Analogously if (y', y) is in the path of minimum length. If (x, x') and (y', y) are in paths of length $p+k$ from x to y, we consider: $[x, x', \alpha_2^s, \ldots, \alpha_p^s, \ldots, \alpha_{p+k-1}^s, y]$ $(\alpha_{p+k-1}^s \neq y')$ and $[x, \alpha_1^t, \ldots, \alpha_p^t, \ldots, \alpha_{p+k-2}^t, y', y]$ $(\alpha_1^t \neq x')$. From these, we construct a path replacing $[x', \alpha_2^s, \ldots, \alpha_p^s]$ with a path of length p, by a path from α_1^t to α_p^s: $[x, \alpha_1^t, \ldots, \alpha_p^s, \ldots, \alpha_{p+k-1}^s, y]$. This new path is disjoint with the other $d-2$ because $[\alpha_p^s, \ldots, \alpha_{p+k-1}^s, y]$ does not intersect any path, and $[\alpha_1^t, \ldots, \alpha_p^s]$ have length p and α_1^t is not used from x by any other path.
If $x = y$ the above construction is also valid.

Proposition 4.3
Let $x, y \in V(KGC(p. d. d^p+1))$ in adjacent partite sets, $x'V(\Gamma^+(x)$ and $y' \in \Gamma^-(y))$. There exist $d-1$ disjoint paths of length at most $2p+1$ from x to y, avoiding the arcs (x, x') and (y', y). Moreover, at most one of them have length $2p+1$.

Proof. By the Prop. 4.1, we know d disjoint paths of length at most $2p$ from x to y. Also, as the out-degree of x and the in-degree of y are d, these paths contain the arcs (x, x') and $(y'. y)$. This situation is the same that in Prop. 4.2, but them minimum path cold be the arc (x, y). For this reason, we study only the case in wich there exist an arc (x, y). By the symmetry of $KGC(p, d, d^p + 1)$ we can assume $x \in V_0$ and $y \in V_1$.

If $y \neq x'$ and $x \neq y'$, then (x, x'), (y', y) are in paths of length $p+1$.
If (x, x') and (y', y) belong to the same path, it is not the minimum one, so we discard it and still have $d-1$ disjoint paths from x to y with length at most $2p$.
If not, we have paths: $[x, x', \alpha_2^s, \ldots, \alpha_p^s, y]$ $(\alpha_p^s \neq y')$ and $[x, \alpha_1^t, \ldots, \alpha_{p-1}^t, y', y]$ $(\alpha_1^s \neq x')$. We discard both and construct a new path replacing the arc (x, x') by a path of length $p+1$, using the vertex α_1^t: $[x, \alpha_1^t, \ldots, x', \alpha_2^s, \ldots, \alpha_p^s, y]$. This path is disjoint with the other $d-2$ because $[x', \alpha_2^s, \ldots, \alpha_p^s, y]$ does not intersect any path, $[\alpha_1^t, \ldots, x']$ have length p and α_1^t is not used from x by any other path.

If $y = x'$ and $x = y'$, then $(x, x') = (y', y) = (x, y)$ and it is enough to discard the arc (x, y).

If $y = x'$ and $x \neq y'$, then $(x, x') = (x, y)$. Here we discard the arc (x, y) and the path by (y', y): $[x, \alpha_1^s, \ldots, \alpha_{p-1}^s, y', y]$ $(\alpha_1^s \neq x')$. We add $d-1$ paths: $[x, \alpha_1^t, \ldots, \alpha_{p-1}^s, y', \alpha_2^t, \ldots, \alpha_p^t, y]$ $(\alpha_p^t \neq y')$. This path is disjoint with the other $d-2$ because $[\alpha_1^s, \ldots, \alpha_{p-1}^s, y]$ does not intersect any path, $[y', \ldots, \alpha_p^t]$ have length p and α_t^1 is not used to arrive to y by any other path.(α_p^t exist because $\Gamma^-(y)$ has d vertices, and we have to avoid x and the ones used by the other paths, which are $d-2$).

If $y \neq x'$ and $x = y'$, then $(x, y) = (y', y)$. Here we discard the arc (x, y) and the path by (x, x'): $[x, x', \alpha_2^s, \ldots, \alpha_p^s, y]$ $(\alpha_p^s \neq y')$. To have $d-1$ path we add: $[x, \alpha_1^t, \ldots, x', \alpha_2^s, \ldots, \alpha_p^s, y]$ $(\alpha_1^t \neq x')$. This path exist and is disjoint with the others by the same arguments we use when $y = x'$ and $x \neq y'$.

Now, we are going to use the above families of paths between vertices of $KGC(p, d, d^p + 1)$ to construct paths in $KGC(p, d, d^{p+k} + d^k)$.

As $KGC(p, d, d^{p+k} + d^k) = L^k(KGC(p, d, d^p + 1))$, we can give for two adjacent vertices u and v the following expressions:

$$u = (x_0, x_1, \ldots, x_k)$$
$$v = (x_1, \ldots, x_k, x_{k+1})$$

with $x_0, \ldots, x_{k+1} \in V(KGC(p, d, d^p + 1))$.

An evident path from u to v is the arc (u, v): $[x_0, x_1, \ldots, x_k, x_{k+1}]$. For the others paths, the general expression is:

$$[x_0, x_1, \ldots, x_k, a_1, a_2, \ldots, a_r, x_1, x_2, \ldots, x_{k+1}]$$

with $r \leq 2p+2$. That is, we need disjoint paths from x_k to x_1 in $KGC(p, d, d^p + 1)$ in the form: $[x_k, a_1, a_2, \ldots, a_r, x_1]$ with $r \leq 2p+2$, and avoiding (x_k, x_{k+1}) and (x_0, x_1) to be disjoint. We are going to use the paths of the above subsection.Taking $x' = x_{k+1}$ and $y' = x_0$ we have:

If $k \equiv_p 0$, $d(x_k, x_1) = 1, p+1$ so they are in adjacent parts of $KGC(p, d, d^p + 1)$, and there are $d-1$ paths from x_k to x_1 of length at most $2p+1$, avoiding the arcs (x_k, x_{k+1}) and (x_0, x_1).

If $k \equiv_p r$, $1 \leq r \leq p-1$, $d(x_k, x_1) = r-1, p-r+1$ so they are in non adjacent sets of $KGC(p, d, d^p + 1)$, and there are $d-1$ paths from x_k to x_1 of length at most $2p+1$, avoiding the arcs (x_k, x_{k+1}) and (x_0, x_1). Moreover, at most one have length $2p+1$.

Now, from a given path $[x_k, a_1, a_2, \ldots, a_r, x_1]$ in the family calculated, then we take a path in the form:

$$[x_0, x_1, \ldots, x_k, a_1, a_2, \ldots, a_r, x_1, x_2, \ldots, x_{k+1}]$$

wich has the conditions for $KGC(p, d, d^{p+k} + d^k)$.
If $1 \leq k \leq p-1$, is not possible to have $k \equiv_p 0$, so x_k and x_1 are not in adjacent partite sets.
If $k = 0$, $KGC(p, d, d^{p+k} + d^k) = KGC(p, d, d^p + 1)$ and the problem is just studied.
We have to prove that the paths induced in $KGC(p, d, d^{p+k} + d^k)$ by the paths constructed in $KGC(p, d, d^p + 1)$ are disjoint. Observe that the format of the path in $KGC(p, d, d^{p+k} + d^k)$ induced by the path $[x_k, a_1, \ldots, a_r, x_1]$ in $KGC(p, d, d^p + 1)$ is $[x_0, x_1, \ldots, x_k, a_1, \ldots, a_r, x_1, \ldots, x_k, x_{k+1}]$. As a consequence, we can work as we do with the $BGC(p, d, d^k)$, proving the independence when the paths have the same length, and extending the result to the general case. Once again, by induction on k we extend the result to the general case:

Theorem 4.4
Let $u, v \in V(KGC(p, d, d^{p+k} + d^k))$. There exist d disjoint paths from u to v with length less or equal than $2p+k+1$. Moreover, at most one of them have length $2p+k+1$.

5. Routing in $BGC(p, d, d^k)$

We are going to use the sets of paths constructed in Section 3 to construct routing algorithms for $BGC(p, d, d^k)$. We refer [3] for a more detailed study about algorithms implementation for fault-tolerant communications.

We assume that before running the routing algorithms, other algorithm was running on the network. These algorithms recognize the faulty elements (nodes and links), giving a list of them as output. Note that this is not a restriction since is the most common way in wich routers work when no acknowledge messages are sent.

As $BGC(p, d, d^{k+1}) = L^k(C_p \otimes K_d^+)$, two given vertices u and v could be represented by sequences:

$$u = (c_1, \ldots, c_r, a_0, a_1, \ldots, a_{k-r}),$$
$$v = (a_0, a_1, \ldots, a_{k-r}, b_1, \ldots, b_r)$$

with all coefficients in $C_p \otimes K_d^+$, and r the distance from u to v.

In the case $r = 1$ we just have a description of the minimum length paths from u to v. In other case, we can consider vertices u' and v' in $BGC(p, d, d^{k-r})$ given by:

$$u' = (c_1, a_0, a_1, \ldots, a_{k-r})$$
$$v' = (a_0, a_1, \ldots, a_{k-r}, b_r)$$

with all coefficients in $C_p \otimes K_d^+$.
Now, $d(u', v') = 1$ and another time, we know the paths from u' to v' and can go from u to v by the paths from u' to v'.

Then, a briefly descrition of the routing algorithm could be:

Input: u, v vertices in $BGC(p, d, d^{k+1})$

- Calculate r, the distance from u to v.
- If $r = 1$ choose the paths of minimum length from the constructed above wich do not intersect the list of faulty nodes.
- If $r \neq 1$ find vertices u', v' as above. Construct paths between u' and v' and extend them to paths from u to v. Choose the one of minimum length from the paths wich does not have any faulty element.

Let us discuss how to implement each item in the algorithm.

To calculate the distance between the input vertices, the most natural way is to compare the sequences that represent them. Once we have done the comparisons we have:

$$r = d(u, v)$$
$$u = c_1, \ldots, c_r, a_1, \ldots, a_{k-r+1}$$
$$v = a_1, \ldots, a_{k-r+1}, b_1, \ldots, b_r$$

Now, as we know the distance, we know also d disjoint paths from u to v. In fact, if they are adjacent, we have a direct construction of them. If not, we take two adjacent vertices:

$$u' = c_r, a_1, \ldots, a_{k-r+1}$$
$$v' = a_1, \ldots, a_{k-r+1}, b_1$$

and from tha paths between them, arise the paths we want.

At this point, we have to choose a path of minimum length in the above set wich not contain neither a faulty node nor a faulty arc.

We can do it in several ways. A first idea could be to construct all paths by increasing order of their lengths, and begin inpecting the shortest until we find one with the conditions desired.

A second option could be to compare the nodes and arcs involved in each path with the faulty ones during its construction. That is, could be not necessary to construct the whole set. So, we can construct one path, check the conditions, and only if it is necessary, we proceed constructing another ones. Also, we can improve this idea, checking the conditions during the construction. That is, in the precise moment we add a node (and obviously an arc) we check that it is not a faulty one. So, we have to take care in not add a faulty element. In the case that we have no other alternative, we discard this construction and begin another.
Also, to make this second option efficient, we have to construct tha paths by increasing order of their lengths.

This is the basic idea we propose. This algorithm must be implementing according to the considerations at the beginning of the section.

Example: Let $d = 7$, $p = 4$ and $k = 5$.

$$BGC(4, 7, 117.649) = L^5(C_4 \otimes K_7^+)$$

Suppose we want to find a path from u to v, with:

$$u = (3,1)(4,6)(1,3)(2,0)(3,2)(4,6)$$
$$v = (1,3)(2,0)(3,2)(4,6)(1,1)(2,3)$$

As $d(u, v) = 2$, we cannot use the direct construction of the paths, so we have to construct them recursively, from paths between vertices at distance 1. Let:

$$u' = (4,6)(1,3)(2,0)(3,2)(4,6),$$
$$v' = (1,3)(2,0)(3,2)(4,6)(1,1)$$

Now, as $d(u', v') = 1$, applying the base construction, we obtain 7 paths from u' to v':

- The arc (u', v'):

$$[(4,6)(1,3)(2,0)(3,2)(4,6)(1,1)]$$

- Paths constructed from paths from $(4,6)$ to $(1,3)$:

$$[(4,6)(1,3)]$$
$$[(4,6)(1,\alpha_1^s)(2,\alpha_2^s)(3,\alpha_3^s)(4,\alpha_4^s)(1,3)]$$

with $\alpha_1^s \neq 1, 3$ and $\alpha_4^s \neq 6$.

These paths give rise to:
$[(4,6)(1,3)(2,0)(3,2)(4,6)(1,3)(2,0)(3,2)(4,6)(1,1)]$
$[u'(1,\alpha_1^s)(2,\alpha_2^s)(3,\alpha_3^s)(4,\alpha_4^s)v']$, with $\alpha_1^s \neq 1, 3$ and $\alpha_4^s \neq 6$.

From these 7 paths from u' to v', by the resursive procedure we obtain the following 7 paths from u to v:

- $[(3,1)(4,6)(1,3)(2,0)(3,2)(4,6)(1,1)(2,3)]$

- $[(3,1)(4,6)(1,3)(2,0)(3,2)(4,6)(1,3)(2,0)$
$(3,2)(4,6)(1,1)(2,3)]$

- $[u(1,\alpha_1^s)(2,\alpha_2^s)(3,\alpha_3^s)(4,\alpha_4^s)v]$, with $\alpha_i^s = 1,\ldots,5$, $\alpha_1^s \neq 1,3$, $\alpha_4^s \neq 6$ and $\alpha_i^s \neq \alpha_i^t$ if $s \neq t$

Now, we have constructed d paths from u to v, and only rest to select the one of minimum (or minimal) length wich do not contain faulty elements.

Let F be the set of faulty nodes and L the set of faulty links. For example, consider:
$F = \{(2,0)(3,2)(4,6)(1,3)(2,0)(3,2),(4,5)(1,4)$
$(2,0)(3,2)(4,5)(1,3)\}$.
$E = \{(3,1)(4,6)(1,3)(2,0)(3,2)(4,6)(1,1)\}$.

The algorithm discard the first two paths and one of the third tipus.

Now, if F and E are:
$F = \{(2,1)(3,2)(4,3)(1,5)(2,0)(3,1),(4,5)(1,4)$
$(2,0)(3,2)(4,5)(1,3)\}$.
$E = \{(2,0)(3,2)(4,6)(1,3)(2,0)(3,2)(4,6)(1,1)$
$(4,6)(1,3)(2,0)(3,2)(4,6)(1,1)(2,3)\}$.

The algorithm discard the second path and two paths and one or two of the third tipus.

6. Routing in $KGC(p,d,d^{p+k}+d^k)$

In the same way we work in Section 5, we are going to define here a routing strategy for $KGC(p,d,d^{p+k}+d^k)$. So, we make use of the paths constructed in Section 4. As there, we assume that we have two list containing faulty nodes and faulty links.

As $KGC(p,d,d^{p+k}+d^k) = L^k(KGC(p,d,d^{p+1}))$, two vertices u and v could be represented by sequences:

$$u = (c_1,\ldots,c_r,a_0,a_1,\ldots,a_{k-r})$$
$$v = (a_0,a_1,\ldots,a_{k-r},b_1,\ldots,b_r)$$

with all coefficients in $KGC(p,d,d^{p+1})$ and r the distance from u to v.

In the case $r=1$ we just have a description of the minimum length paths from u to v. In other case, we can consider vertices u' and v' in $KGC(p,d,d^{p+k}+d^k)$ given by:

$$u' = (c_1,a_0,a_1,\ldots,a_{k-r})$$
$$v' = (a_0,a_1,\ldots,a_{k-r},b_r)$$

with all coefficients in $KGC(p,d,d^{p+1})$.

Now, $d(u',v')=1$ and another time, we know the paths from u' to v' and can go from u to v by the paths from u' to v'.

Then, a briefly descrition of the routing algorithm could be:

Given two vertices u,v in $KGC(p,d,d^{p+k}+d^k)$:

- Calculate r, the distance from u to v.

- If $r=1$ choose the paths of minimum length from the constructed above wich do not intersect the list of faulty nodes.

- If $r \neq 1$ find vertices u',v' as above. Construct paths between u' and v' and extend them to paths from u to v. Choose the one of minimum length from the paths wich not intersect the list of faulty nodes.

That is, the routing strategy is the same that for $BGC(p,d,d^k)$. Again, the considerations to implement the routing algorithm are the same.

Example: Let take $d=4$, $p=5$ and $k=6$.

$$KGC(5,4,1025) = L^4(C_5 \otimes GK(4,1025))$$

We want a path from u to v, with: $u = (5,77)(1,715)(2,814)(3,182)(4,297)$
$(5,170)(1,681)$
$v = (3,182)(4,297)(5,170)(1,681)(2,860)$
$(3,660)(4,435)$

As $d(u,v)=3$, we have to construct the paths recursively, from paths between vertices at distance 1. So, we determine:

$$u' = (2,814)(3,182)(4,297)(5,170)(1,681)$$
$$v' = (3,182)(4,297)(5,170)(1,681)(2,860)$$

Now, as $d(u',v')=1$, applying the base construction, we obtain 5 paths from u' to v':

- The arc (u',v'):
$[(2,814)(3,182)(4,297)(5,170)(1,681)(2,860)]$.

- Paths from paths between $(1,681)$ and $(3,182)$:
$[(1,681)(2,\alpha_2^{s_1})(3,\alpha_3^{s_1})(4,\alpha_4^{s_1})(5,\alpha_5^{s_1})$
$(1,\alpha_1^{s_2})(2,\alpha_2^{s_2})(3,182)]$
giving rise to:
$[u'(2,\alpha_2^{s_1})(3,\alpha_3^{s_1})(4,\alpha_4^{s_1})(5,\alpha_5^{s_1})(1,\alpha_1^{s_2})(2,\alpha_2^{s_2})v']$

From these 5 paths from u' to v', by the resursion we obtain the following 4 paths from u to v:

- If $((1,681),(2,860))$,$((2,814),(3,182))$ are in the same path, we discard it.

- If $((1,681),(2,860))$,$((2,814),(3,182))$ are in different paths, we discard both, and add the path obtaining by replacing the arc $((1,681),(2,860))$ by a cycle of length p, in the path that contain it.

$$[u(2,\alpha_2^t)(3,\alpha_3^t)(4,\alpha_4^t)(5,\alpha_5^t)(1,\alpha_1^t)(2,860)$$
$$(3,\alpha_3^{s_1})(4,\alpha_4^{s_1})(5,\alpha_5^{s_1})(1,\alpha_1^{s_2})(2,\alpha_2^{s_2})v]$$

Now, we have constructed d paths from u to v, and only rest to select the one of minimum (or minimal) length wich do not contain faulty elements, according with the considerations in the above section.

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