

Edge Connectivity of Iterated P_3 -Path Graphs

Daniela Ferrero

Department of Mathematics
Southwest Texas State University
San Marcos, TX 78666, USA

Abstract

For a given graph G and a positive integer k the P_k -path graph, $P_k(G)$, has for vertices the set of all paths of length k in G . Two vertices are adjacent when the intersection of the corresponding paths forms a path of length $k - 1$ in G , and their union forms either a cycle or a path of length $k + 1$ in G . Path graphs were proposed as an extension of line graphs. Indeed, P_1G coincides with the line graph of G . In this paper we study the connectivity and superconnectivity of P_3 -path graphs and more generally, of iterated P_3 -path graphs.

1 Introduction

The P_k -path graph corresponding to a graph G has for vertices the set of all paths of length k in G . Two vertices are connected by an edge whenever the intersection of the corresponding paths forms a path of length $k - 1$ in G , and their union forms either a cycle or a path of length $k + 1$ in G . Intuitively, this means that the vertices are adjacent if and only if one can be obtained from the other by shifting the corresponding paths in G . Path graphs were introduced by Broersma and Hoede in [2] as a natural generalization of line graphs. A characterization of P_2 -path graphs is given in [2] and [7], and distance properties of path graphs are studied in [1] and [5]. The connectivity and of path graphs was studied by Knor, Niepel [4, 6] and Mallah [6]. Note that the path graph can be thought of as an operator on graphs, and therefore, we can study graphs arising from the iteration of the P_k -path graph operator.

2 Definitions, notation and previous results

Let $G = (V, E)$ be a simple graph, i.e. with no loops or multiple edges, with vertex set $V(G)$ and edges $E(G)$. The *neighbourhood* of a vertex v is the set $N_G(v)$, of all vertices adjacent to v . The *degree* of a vertex v is $\deg(v) = |N_G(v)|$. The *minimum degree* of the graph G , $\delta(G)$, is the minimum degree over all vertices of G .

A graph G is called *connected* if every pair of vertices is joined by a path. An *edge cut* in a graph G is a set A of edges of G such that $G - A$ is not connected. Note that if A is a minimal edge cut of a connected graph G , then $G - A$ necessarily has exactly two connected components. Hence we denote the edge cut A as $A = (C, \overline{C})$, where C is a proper subset of $V(G)$, \overline{C} denotes its complement, and (C, \overline{C}) denotes the set of edges in A . The *edge-connectivity* $\lambda(G)$ of a graph G is the cardinality of a minimal edge cut of G . Since $\lambda(G) \leq \delta(G)$, a graph G is said to be *maximally edge-connected* when $\lambda(G) = \delta(G)$. A minimum edge cut (C, \overline{C}) is called trivial if $C = \{v\}$ or $\overline{C} = \{v\}$ for some vertex v with $\deg(v) = \delta(G)$. A maximally edge-connected graph is called *super- λ* if every edge cut (C, \overline{C}) of cardinality $\delta(G)$ is trivial. The superconnectivity of a graph is denoted by $\lambda_1(G)$ and it is defined as $\lambda_1(G) = \min\{|(C, \overline{C})|, (C, \overline{C}) \text{ is a non trivial edge cut}\}$. Then, a graph G is super- λ if and only if $\lambda_1(G) > \delta(G)$.

The *girth* of a graph G is denoted by $g(G)$ and it represents the length of a shortest cycle in G .

The P_k -path graph corresponding to a graph G has for vertices the set of all paths of length k in G . Two vertices are connected by an edge whenever the intersection of the corresponding paths forms a path of length $k - 1$ in G , and their union forms either a cycle or a path of length $k + 1$ in G . The connectivity of P_k path graphs was previously studied by Knor and Niepel [4]. They introduced some notation to formulate two important theorems. Next we recall those concepts and results.

Let P_3^0 denotes a subgraph of G induced by the vertices in a path of length 3, say v_0, v_1, v_2, v_3 , such that neither v_0 nor v_3 has a neighbor in $V(G) - \{v_1, v_2\}$. A path A is in P_3^0 if and only if $A = v_0, v_1, v_2, v_3$. Analogously, P_4^0 denotes an induced subgraph of G with a path of length x, v_0, v_1, v_2, y in which every neighbor of v_0 and v_2 except v_0, v_1 and v_2 has degree 1, or it has degree 2 and in this case it is adjacent to v_1 . Moreover, no vertex of $V(P_4^0) - \{v_1\}$ is adjacent to a vertex of $V(G) - V(P_4^0)$ in G . A path A of length 3 is in P_4^0 if v_0, v_1, v_2 is a subpath of A .

For an independent set of vertices S , let K_4^* denote a graph obtained from $K_4 \cup S$ by joining all vertices of S to one special vertex of K_4 .

Let $K_{2,t}$ be a complete bipartite graph and let (X, Y) be a bipartition of $K_{2,t}$ where $X = \{v_1, v_2\}$. Join t sets of independent vertices by

edges, each to one vertex of Y ; further, glue a set of stars with at least 3 vertices by one endvertex, each to v_1 or to v_2 ; glue a set of triangles by one vertex, each either to v_1 or v_2 ; and finally, join v_1 to v_2 by an edge. The resulting graph is denoted by $K_{2,t}^*$.

Theorem A [4] Let G be a connected graph such that $P_3(G)$ is not empty. Then, $P_3(G)$ is disconnected if and only if one of the following conditions holds

- 1) G contains a P_t^0 , $y \in \{3, 4\}$, and a path A of length 3 such that $A \notin P_t^0$
- 2) G is isomorphic to K_4^*
- 3) G is isomorphic to $K_{2,t}^*$, $t \geq 1$. ■

Given a simple graph G and a path of length k in it, let us say $u_0u_1 \dots u_k$, clearly that path determines a vertex in P_kG . We are going to denote the vertex in P_kG by $U = u_0u_1 \dots u_k$ and the path in G by $U : u_0, u_1, \dots, u_k$.

The reader is referred to [3] for additional concepts and results about graphs.

3 Edge connectivity of P_3 -path graphs

To measure the connectivity of a graph is obviously interesting when the graph is connected. For this reason, the foundation of this section is Theorem A, which provides a characterization of connected P_3 -path graphs. However, we are going to work with a subclass of graphs G for which P_3G is connected. Notice that the conditions on Theorem A are not preserved under the iteration of the P_3 -path graph operator. We are going to show conditions on the minimum degree that guarantee connectivity, which are preserved under the P_3 -path graph operator. The next lemmas provide some technical results needed for that purpose.

Lemma 3.1 *Let G be a simple graph with minimum degree $\delta \geq 3$, and let $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$ be two vertices in P_3G . If $|\{a_0, a_1, a_2, a_3\} \cap \{b_0, b_1, b_2, b_3\}| = 1$, there is path in P_3G joining $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$.*

Proof: First, notice that it is enough to consider the cases in which $\{a_0, a_1, a_2, a_3\} \cap \{b_0, b_1, b_2, b_3\} = a_0$ or a_1 . Also, if $\{a_0, a_1, a_2, a_3\} \cap \{b_0, b_1, b_2, b_3\} = a_0$, we have four possible situations, $a_0 = b_0$, $a_0 = b_1$, $a_0 = b_2$ or $a_0 = b_3$. In the cases $a_0 = b_0$ or $a_0 = b_3$, it is clear that there exists a path joining $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$, so it is only necessary to solve the cases $a_0 = b_1$, $a_0 = b_2$.

If $a_0 = b_1$, since $\delta \geq 3$ we can consider a vertex $c \in N^+(b_3) - \{b_1, b_2\}$, and the path in P_3G , $a_0a_1a_2a_3, b_2a_0a_1a_2, b_3b_2a_0a_1, cb_3b_2b_1, b_3b_2b_1b_0$.

If $a_0 = b_2$, since $\delta \geq 3$ we can consider a vertex $c \in N^+(b_0) - \{b_1, b_2\}$, and the path in P_3G , $a_0a_1a_2a_3, b_1a_0a_1a_2, b_0b_1a_0a_1, cb_0b_1b_2, b_0b_1b_2b_3$. If $\{a_0, a_1, a_2, a_3\} \cap \{b_0, b_1, b_2, b_3\} = a_1$, we have again four possible situations, $a_1 = b_0$, $a_1 = b_1$, $a_1 = b_2$ or $a_1 = b_3$, but the case $a_1 = b_0$ is analogous to the case $a_0 = b_1$ and $a_1 = b_3$ is analogous to the case $a_0 = b_2$ that were solved previously.

If $a_1 = b_1$, since $\delta \geq 3$ we can consider vertices $c \in N^+(a_3) - \{a_1, a_2\}$ and $d \in N^+(b_3) - \{b_1, b_2\}$, and then the path in P_3G , $a_0a_1a_2a_3, a_1a_2a_3c, b_2b_1a_2a_3, b_3b_2b_1a_2, db_3b_2b_1, b_3b_2b_1b_0$.

If $a_1 = b_2$, since $\delta \geq 3$ we can consider vertices $c \in N^+(a_3) - \{a_1, a_2\}$ and $d \in N^+(b_0) - \{b_1, b_2\}$, and then the path in P_3G , $a_0a_1a_2a_3, a_1a_2a_3c, b_1a_1a_2a_3, b_0b_1b_2a_2, db_0b_1b_2, b_0b_1b_2b_3$. ■

Lemma 3.2 *Let G be a simple graph with minimum degree $\delta \geq 4$, and let $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$ be two vertices in P_3G . If $|\{a_0, a_1, a_2, a_3\} \cap \{b_0, b_1, b_2, b_3\}| = 2$, there is path in P_3G joining $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$.*

Proof: In order to give paths between the vertices $A = a_0a_1a_2a_3$ and $B = b_0b_1b_2b_3$ it is sufficient to consider the next different cases.

i) If $a_0, a_1 \in \{b_0, b_1, b_2, b_3\}$ we have the following paths depending on B

- $B = a_0a_1b_2b_3$: $A, ca_0a_1a_2, dca_0a_1, ca_0a_1b_2, B$ with $c \in N(a_0) - \{a_1, a_2, b_2\}, d \in N(c) - \{a_0, a_1\}$
- $B = a_0b_1a_1b_3$: $A, a_1a_2a_3c, b_1a_1a_2a_3, a_0b_1a_1a_2, ea_0b_1a_1, B$ with $c \in N(a_3) - \{a_1, a_2\}, e \in N(a_0) - \{b_1, a_1\}$
- $B = a_0b_1b_2a_1$: $A, b_1a_0a_1a_2, b_2b_1a_0a_1, B$
- $B = a_1a_0b_2b_3$: $A, b_2a_0a_1a_2, B$
- $B = b_0a_0a_1b_3$: $A, b_0a_0a_1a_2, cb_0a_0a_1, B$ with $c \in N(b_0) - \{a_0, a_1\}$
- $B = b_0a_0b_2a_1$: $A, b_0a_0a_1a_2, cb_0a_0a_1, dcb_0a_0, cb_0a_0b_2, B$ with $c \in N(b_0) - \{a_0, a_1, b_2\}$ and $d \in N(c) - \{b_0, a_0\}$

ii) If $a_1, a_2 \in \{b_0, b_1, b_2, b_3\}$ we have the following paths depending on B

- $B = a_1a_2b_2b_3$: $A, ca_0a_1a_2, a_0a_1a_2b_2, B$ with $c \in N(a_0) - \{a_1, a_2\}$
- $B = a_1b_1a_2b_3$: $A, ca_0a_1a_2, dca_0a_1, ca_0a_1b_1, a_0a_1b_1a_2, B$ with $c \in N(a_0) - \{a_1, a_2, b_1\}, d \in N(c) - \{a_0, a_1\}$
- $B = a_1b_1b_2a_2$: $A, ca_0a_1a_2, a_0a_1a_2b_2, a_1a_2b_2b_1, B$ with $c \in N(a_0) - \{a_1, a_2\}$
- $B = a_2a_1b_2b_3$: $A, a_1a_2a_3e, b_2a_1a_2a_3, B$ with $e \in N(a_3) - \{a_1, a_2\}$

$$\begin{aligned}
B = b_0a_1a_2b_3: & A, a_1a_2a_3e, b_0a_1a_2a_3, cb_0a_1a_2, B \text{ with} \\
& e \in N(a_3) - \{a_1, a_2\}, c \in N(b_0) - \{a_1, a_2\} \\
B = b_0a_1b_2a_3: & A, a_1a_2a_3e, a_2a_3ef, b_2a_2a_3e, a_1b_2a_2a_3, B \text{ with} \\
& e \in N(a_3) - \{a_1, a_2, b_2\}, f \in N(e) - \{a_2, a_3\}
\end{aligned}$$

iii) If $a_0, a_3 \in \{b_0, b_1, b_2, b_3\}$ we have the following paths depending on B

$$\begin{aligned}
B = a_0a_3b_2b_3: & A, a_3a_0a_1a_2, b_2a_3a_0a_1, B \\
B = a_0b_1a_3b_3: & A, b_1a_0a_1a_2, a_3b_1a_0a_1, B \\
B = a_0b_1b_2a_3: & A, b_1a_0a_1a_2, b_2b_1a_0a_1, B \\
B = a_3a_0b_2b_3: & A, a_1a_2a_3a_0, a_2a_3a_0b_2, B \\
B = b_0a_0a_3b_3: & A, a_1a_2a_3a_0, a_2a_3a_0b_0, a_3a_0b_0c, B \text{ with} \\
& c \in N(b_0) - \{a_3, a_0\} \\
B = b_0a_0b_2a_3: & A, a_1a_2a_3b_2, a_2a_3b_2a_0, B
\end{aligned}$$

iv) If $a_0, a_2 \in \{b_0, b_1, b_2, b_3\}$ we have the following paths depending on B

$$\begin{aligned}
B = a_0a_2b_2b_3: & A, ca_0a_1a_2, a_0a_1a_2b_2, a_1a_2b_2b_3, a_2b_2b_3e, B \\
& c \in N(a_0) - \{a_1, a_2\}, e \in N(b_3) - \{a_2, b_2\} \\
B = a_0b_1a_2b_3: & A, b_1a_0a_1a_2, a_2b_1a_0a_1, B \\
B = a_0b_1b_2a_2: & A, b_1a_0a_1a_2, b_2b_1a_0a_1, B \\
B = a_2a_0b_2b_3: & A, b_2a_0a_1a_2, b_3b_2a_0a_1, eb_3b_2a_0, B \text{ with} \\
& e \in N(b_3) - \{b_2, a_0\} \\
B = b_0a_0a_2b_3: & A, b_0a_0a_1a_2, cb_0a_0a_1, dcb_0a_0, cb_0a_0a_2, B \text{ with} \\
& c \in N(b_0) - \{a_0, a_1, a_2\}, d \in N(c) - \{b_0, a_0\} \\
B = b_0a_0b_2a_2: & A, b_0a_0a_1a_2, a_0a_1a_2b_2, a_1a_2b_2a_0, B. \blacksquare
\end{aligned}$$

The condition on the degree necessary in the previous lemma can be relaxed if the graph does not have triangles, so we can state the following result, which can be proved in a similar way.

Lemma 3.3 *Let G be a simple graph with no triangles and with minimum degree $\delta \geq 3$, and let $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$ be two vertices in P_3G . If $|\{a_0, a_1, a_2, a_3\} \cap \{b_0, b_1, b_2, b_3\}| = 2$, there is path in P_3G joining $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$. ■*

Lemma 3.4 *Let G be a simple graph with minimum degree $\delta \geq 4$, and let $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$ be two vertices in P_3G . If $|\{a_0, a_1, a_2, a_3\} \cap \{b_0, b_1, b_2, b_3\}| = 3$, there is path in P_3G joining $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$.*

Proof: In order to give paths between the vertices $A = a_0a_1a_2a_3$ and $B = b_0b_1b_2b_3$ it suffices to consider the next different cases.

i) If $a_0 \notin \{b_0, b_1, b_2, b_3\}$ we have the following paths, depending on B

$$B = b_0a_1a_2a_3: A, a_1a_2a_3c, B, \text{ with } c \in N(a_3) - \{a_1, a_2\}$$

$$B = a_1b_0a_2a_3: A, a_1a_2a_3c, a_2a_3cd, b_0a_2a_3c, B, \text{ with } c \in N(a_3) - \{a_1, a_2, b_0\}, d \in N(c) - \{a_2, a_3\}$$

$$B = a_1a_2b_0a_3: A, ca_0a_1a_2, a_0a_1a_2b_0, B, \text{ with } c \in N(a_0) - \{a_1, a_2\}$$

$$B = a_1a_2a_3b_0: A, B$$

$$B = b_0a_1a_3a_2: A, a_1a_2a_3e, b_0a_1a_2a_3, cb_0a_1a_2, dcb_0a_1, cb_0a_1a_3, B, \text{ with } e \in N(a_3) - \{a_1, a_2\}, c \in N(b_0) - \{a_1, a_2, a_3\}, d \in N(c) - \{b_0, a_1\}$$

$$B = a_1b_0a_3a_2: A, ca_0a_1a_2, dca_0a_1, ca_0a_1b_0, a_0a_1b_0a_3, B, \text{ with } c \in N(a_0) - \{a_1, a_2, b_0\}, d \in N(c) - \{a_0, a_1\}$$

$$B = a_1a_3b_0a_2: A, ca_0a_1a_2, dca_0a_1, ca_0a_1a_3, a_0a_1a_3b_0, B, \text{ with } c \in N(a_0) - \{a_1, a_2, a_3\}, d \in N(c) - \{a_0, a_1\}$$

$$B = a_1a_3a_2b_0: A, ca_0a_1a_2, dca_0a_1, ca_0a_1a_3, a_0a_1a_3a_2, B, \text{ with } c \in N(a_0) - \{a_1, a_2, a_3\}, d \in N(c) - \{a_0, a_1\}$$

$$B = b_0a_2a_1a_3: A, ca_0a_1a_2, a_0a_1a_2b_0, a_1a_2b_0e, B, \text{ with } c \in N(a_0) - \{a_1, a_2\}, e \in N(b_0) - \{a_1, a_2\}$$

$$B = a_2b_0a_1a_3: A, a_1a_2a_3e, a_2a_3ef, b_0a_2a_3e, a_1b_0a_2a_3, B, \text{ with } e \in N(a_3) - \{a_1, a_2, b_0\}, f \in N(e) - \{a_2, a_3\}$$

$$B = a_2a_1b_0a_3: A, a_1a_2a_3b_0, a_2a_3b_0a_1, B$$

$$B = a_2a_1a_3b_0: A, a_1a_2a_3b_0, a_2a_3b_0e, a_3b_0ef, a_1a_3b_0e, B, \text{ with } e \in N(b_0) - \{a_1, a_2, a_3\}, f \in N(e) - \{b_0, a_3\}$$

ii) If $a_1 \notin \{b_0, b_1, b_2, b_3\}$ we have the following paths, depending on B

$$B = b_1a_0a_2a_3: A, a_1a_2a_3c, a_2a_3cd, a_0a_2a_3c, B, \text{ with } c \in N(a_3) - \{a_1, a_2, a_0\}, d \in N(c) - \{a_2, a_3\}$$

$$B = a_0b_1a_2a_3: A, a_1a_2a_3c, a_2a_3cd, b_1a_2a_3c, B, \text{ with } c \in N(a_3) - \{a_1, a_2, b_1\}, d \in N(c) - \{a_2, a_3\}$$

$$B = a_0a_2b_1a_3: A, ca_0a_1a_2, a_0a_1a_2b_1, a_1a_2b_1a_3, a_2b_1a_3e, B, \text{ with } c \in N(a_0) - \{a_1, a_2, a_0\}, e \in N(a_3) - \{a_2, b_1\}$$

$$B = a_0a_2a_3b_1: A, a_1a_2a_3b_1, a_2a_3b_1e, B, \text{ with } e \in N(b_1) - \{a_2, a_3\}$$

$$B = b_1a_0a_3a_2: A, b_1a_0a_1a_2, cb_1a_0a_1, dcb_1a_0, cb_1a_0a_3, B, \text{ with } c \in N(b_1) - \{a_0, a_1, a_3\}, d \in N(c) - \{b_1, a_0\}$$

$$B = a_0b_1a_3a_2: A, a_2a_1a_0b_1, a_1a_0b_1a_3, B$$

$$B = a_0a_3b_1a_2: A, a_2a_1a_0a_3, a_1a_0a_3b_1, B$$

$$B = a_0a_3a_2b_1: A, a_3a_0a_1a_2, a_2a_3a_0a_1, B$$

$$B = b_1a_2a_0a_3: A, ca_0a_1a_2, a_0a_1a_2b_1, a_1a_2b_1e, a_2b_1ef, a_0a_2b_1e, B \text{ with} \\ c \in N(a_0) - \{a_1, a_2, a_0\}, e \in N(b_1) - \{a_2, a_1, a_0\}, \\ f \in N(e) - \{a_2, b_1\}$$

$$B = a_2b_1a_0a_3: A, a_1a_2a_3a_0, a_2a_3a_0b_1, B$$

$$B = a_2a_0b_1a_3: A, a_1a_2a_3b_1, a_2a_3b_1a_0, B$$

$$B = a_2a_0a_3b_1: A, a_1a_2a_3b_1, a_2a_3b_1e, a_3b_1ef, a_0a_3b_1e, B, \text{ with} \\ e \in N(b_1) - \{a_2, a_3, a_0\}, f \in N(e) - \{a_3, b_1\}. \blacksquare$$

As with Lemma 3.2, for Lemma 3.4 we can obtain the following improvement for graphs without triangles.

Lemma 3.5 *Let G be a simple graph with no triangles and with minimum degree $\delta \geq 3$, and let $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$ be two vertices in P_3G . If $|\{a_0, a_1, a_2, a_3\} \cap \{b_0, b_1, b_2, b_3\}| = 3$, there is path in P_3G joining $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$. \blacksquare*

Lemma 3.6 *Let G be a simple graph with minimum degree $\delta \geq 4$, and let $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$ be two vertices in P_3G . If $|\{a_0, a_1, a_2, a_3\} \cap \{b_0, b_1, b_2, b_3\}| = 4$, there is path in P_3G joining $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$.*

Proof: In order to prove this statement we are going to give a path between the vertex $U = a_0a_1a_2a_3$ and any other vertex V obtained by a permutation on the set $\{a_0, a_1, a_2, a_3\}$. However, notice that since we consider undirected paths, among the 23 possibilities for the second vertex, there are only 11 different cases. Furthermore, since U is adjacent to $a_3a_0a_1a_2$ and $a_2a_3a_0a_1$, and it is at distance 2 of $a_2a_3a_0a_1$, it only remains to study eight cases that can be reduced to the following situations.

If $V = a_0a_3a_1a_2$ (or $V = a_1a_2a_0a_3$), since $\delta \geq 4$ we can consider vertices $c \in N(a_0) - \{a_1, a_2, a_3\}$ and $d \in N(c) - \{a_0, a_3\}$, and then the path in P_3G , $U, a_1a_2a_3a_0, a_2a_3a_0c, a_3a_0cd, a_1a_3a_0c, V$.

If $V = a_0a_2a_3a_1$ (or $V = a_2a_0a_1a_3$), since $\delta \geq 4$ we can consider vertices $c \in N(a_0) - \{a_1, a_2\}$ and $d \in N(c) - \{a_0, a_1\}$, and then the path in P_3G , $U, ca_0a_1a_2, dca_0a_1, ca_0a_1a_3, a_0a_1a_3a_2, V$.

If $V = a_0a_1a_3a_2$ (or $V = a_1a_0a_2a_3$), since $\delta \geq 4$ we can consider vertices $c \in N(a_0) - \{a_1, a_2, a_3\}$ and $d \in N(c) - \{a_0, a_1\}$, and then the path in P_3G , $U, ca_0a_1a_2, dca_0a_1, ca_0a_1a_3, V$.

If $V = a_1a_3a_0a_2$, since $\delta \geq 4$ we can consider vertices $c \in N(a_0) - \{a_2, a_3, a_1\}$ and $d \in N(c) - \{a_0, a_3\}$, and then the path in P_3G , $U, a_1a_2a_3a_0, a_2a_3a_0c, a_3a_0cd, a_1a_3a_0c, a_2a_1a_3a_0, V$.

If $V = a_3a_1a_2a_0$, since $\delta \geq 4$ we can consider vertices $e \in N(a_3) - \{a_2, a_1\}$, $c \in N(a_1) - \{a_0, a_2, a_3\}$ and $d \in N(c) - \{a_0, a_1\}$, and then the path in P_3G , $U, a_1a_2a_3e, ca_1a_2a_3, dca_1a_2, ca_1a_2a_0, a_1a_2a_0a_3, V$.

The cases between parenthesis can be solved analogously but considering vertices $c \in N(a_3)$ and $f \in N(e)$ with the corresponding conditions to guarantee that the sequences involving them do not repeat vertices of g so they represent vertices in P_3G . ■

Once again, for graphs without triangles there is an improvement of the previous lemma.

Lemma 3.7 *Let G be a simple graph with no triangles and with minimum degree $\delta \geq 3$, and let $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$ be two vertices in P_3G . If $|\{a_0, a_1, a_2, a_3\} \cap \{b_0, b_1, b_2, b_3\}| = 4$, there is path in P_3G joining $a_0a_1a_2a_3$ and $b_0b_1b_2b_3$.*

Theorem 3.8 *Let G be a connected graph with minimum degree $\delta \geq 4$. Then, P_3G is connected.*

Proof: Let $U = a_0a_1a_2a_3$ and $V = b_0b_1b_2b_3$ be two different vertices in P_3G . If $|\{a_0, a_1, a_2, a_3\} \cap \{b_0, b_1, b_2, b_3\}| = \emptyset$, since G is connected there is a path joining the endpoints of the path $U : a_0, a_1, a_2, a_3$ and the endpoints of the path $V : b_0, b_1, b_2, b_3$. Let us see that such path induces a path between U and V in P_3G . Indeed, if there is a path joining an endpoint of U and an endpoint of V which does not contain internal vertices neither in U nor in V , it is easy to prove that such path in G induces a path between the vertices U and V in P_3G . Otherwise, there is for example, a path joining a_0 with b_1 or b_2 , which does not contain internal vertices in U or in V . Let us assume that the path joins a_0 and b_2 . Then, since $\delta \geq 4$ there exists a vertex $c \in N(b_0) - \{b_1, b_2\}$ and as a consequence, a vertex $cb_0b_1b_2$ which is adjacent with the vertex V . From such vertex it is possible to reach the vertex U using the path between b_2 and a_0 . Analogously it can be solved the case in which there is a path between a_0 and b_1 . If $|\{a_0, a_1, a_2, a_3\} \cap \{b_0, b_1, b_2, b_3\}| \neq \emptyset$, then $|\{a_0, a_1, a_2, a_3\} \cap \{b_0, b_1, b_2, b_3\}| = 1, 2, 3$ or 4 , and we can respectively apply lemmas 3.1, 3.2, 3.4 or 3.6 to guarantee the existence of a path joining U and V . ■

Note that the above theorem establishes a sufficient condition for a connected graph G to have a connected path graph P_3G . This result is used in this section because the condition $\delta \geq 4$ is preserved under the P_3 -path graph operator. However, Theorem A provides a stronger result.

Corollary 3.9 *Let G be a connected graph with minimum degree $\delta \geq 4$. Then, for every positive integer n , P_3^nG is connected.*

Proof: First, notice that the minimum degree of P_3G is lower bounded by $2(\delta - 2)$, so more generally, the minimum degree of P_3^nG is lower bounded by $2^n\delta - 2^{n+2} + 4$, which is at least 4. Then, it is enough to reason by induction on n and apply Theorem 3.8 to get the result. ■

As it was done in Theorem 3.8, but using Lemmas 3.3, 3.5 and 3.7 instead of Lemmas 3.2, 3.4 and 3.6 we obtain the following result for graphs without triangles.

Theorem 3.10 *Let G be a connected graph with no triangles and with minimum degree $\delta \geq 3$. Then, P_3G is connected. ■*

The hypothesis of the previous theorem is preserved under the P_3 -path graph operator, so there is the following corollary.

Corollary 3.11 *Let G be a connected graph with no triangles and with minimum degree $\delta \geq 3$. Then, for every positive integer n , P_3^nG is connected. ■*

Now that we have found a class of graphs for which the P_3 -path graph operator gives connected graphs we shall measure the connectivity of that class.

Lemma 3.12 *Let G be a connected graph with minimum degree $\delta \geq 4$. Then, there exist $\delta - 3$ disjoint paths of length 3 between any two adjacent vertices in P_3G .*

Proof: Without loss of generality, we can assume that two adjacent vertices in P_3G can be written as $U = u_0u_1u_2u_3$ and $V = u_1u_2u_3u_4$. Since $\delta \geq 4$, there exist $b_i \in N_G(u_3) - \{u_1, u_2, u_4\}$, $i = 1, \dots, \delta - 3$ and $c_j \in N_G(u_1) - \{u_0, u_2, u_3\}$, $j = 1, \dots, \delta - 3$, and as a consequence, there exist vertices $u_1 \dots u_k b_i$ and $c_j u_1 \dots u_k$ in P_3G . Then, because of the adjacency rules in P_3G , for each $i = 1, \dots, \delta - 3$ we can assign each b_i with one particular c_j , that for simplicity will be denoted as c_i . Then, there is a path $\mathcal{P}_i : U, u_1u_2u_3b_i, c_iu_1u_2u_3, V$ in P_3G , which obviously has length 3 and joins U and V . Let us see that these paths are disjoint. In fact, since the only case in which two paths \mathcal{P}_i and \mathcal{P}_j could share a vertex different from the endvertices of (U, V) is if $u_1u_2u_3b_i = c_ju_1u_2u_3$, which implies in particular $u_2 = u_1$, which is not possible because u_0, u_1, u_2, u_3 is a simple path in G . Therefore, we have obtained the $\delta - 3$ paths claimed. ■

Lemma 3.13 *Let G be a connected graph with no triangles and with minimum degree $\delta \geq 3$. Then, there exist $\delta - 2$ disjoint paths of length 3 between any two adjacent vertices in P_3G .*

Proof: Let $U = u_0u_1u_2u_3$ and $V = u_1u_2u_3u_4$ be two adjacent vertices in P_3G . Since $\delta \geq 3$, there exist $b_i \in N_G(u_3) - \{u_2, u_4\}$, $i = 1, \dots, \delta - 2$ and $c_j \in N_G(u_1) - \{u_0, u_2\}$, $j = 1, \dots, \delta - 2$, and as a consequence, there exist vertices $u_1 \dots u_k b_i$ and $c_j u_1 \dots u_k$ in P_3G . Since G does not have triangles, assigning each vertex b_i with one particular vertex c_i , for every $i = 1, \dots, \delta - 2$ there is a path $\mathcal{P}_i : u, u_1u_2u_3b_i, c_iu_1u_2u_3, v$ in P_3G , which obviously has length 3 and joins U and V . Those paths are disjoint since the only case in which two paths \mathcal{P}_i and \mathcal{P}_j could share a vertex different from the endvertices of (U, V) is if $u_1u_2u_3b_i = c_ju_1u_2u_3$, which implies in particular $u_2 = u_1$, which is not possible because u_0, u_1, u_2, u_3 is a simple path in G . ■

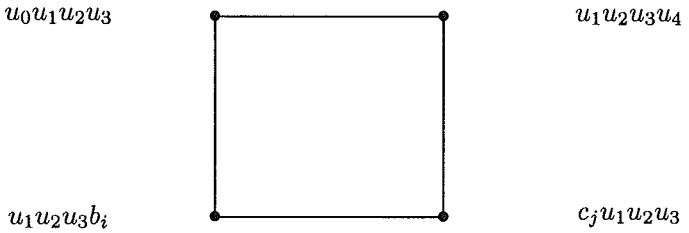


Fig.1 Paths given in Lemma 3.12 and Lemma 3.13

Observe that the P_3 -path graph operator transforms cycles of length 3 into cycles of length 3. As a consequence, the previous lemmas give rise to the following corollary, which illustrates an interesting property of P_3 -path graphs.

Corollary 3.14 *Let G be a connected graph with minimum degree $\delta \geq 3$. Then, $g(P_3G) = 3$ or 4. Moreover, $g(P_3G) = 3$ if and only if $g(G) = 3$. ■*

Lemma 3.15 *Let G be a connected graph with minimum degree $\delta \geq 4$. Then, there exist $\delta - 2$ edge disjoint paths of length 8 between any two adjacent vertices in P_3G .*

Proof: Without loss of generality, we can assume that two adjacent vertices in P_3G can be written as $U = u_0u_1u_2u_3$ and $V = u_1u_2u_3u_4$. Since $\delta \geq 4$, for each $i = 1, \dots, \delta - 2$ there exist vertices

$$\begin{aligned} a_1^i &\in N_G(u_0) - \{u_1, u_2\} \\ a_2^i &\in N_G(u_2) - \{u_0, u_1\} \\ a_3^i &\in N_G(a_2^i) - \{u_1, u_2\} \\ a_4^i &\in N_G(a_3^i) - \{u_2, a_2^i\} \\ a_5^i &\in N_G(u_4) - \{u_2, u_3\} \end{aligned}$$

Also, if $a_2^i \neq u_3$, for each $i = 1, \dots, \delta - 2$ we have a path Q_i

$$U, a_1^i u_0 u_1 u_2, u_0 u_1 u_2 a_2^i, u_1 u_2 a_2^i a_3^i, u_2 a_2^i a_3^i a_4^i, \\ u_3 u_2 a_2^i a_3^i, u_4 u_3 u_2 a_2^i, u_2 u_3 u_4 a_5^i, V$$

These paths are edge disjoint because for each value of $i = 1, \dots, \delta - 2$ we can choose different vertices $a_1^i, a_3^i, a_4^i, a_5^i$, so even when for a_2^i there are only $\delta - 3$ possible values because we need the condition $a_2^i \neq u_3$, using the fact that $u_i \neq u_j$ if $i \neq j$ it can be shown that the paths can share at most one vertex, so they are edge disjoint. ■

Lemma 3.16 *Let G be a connected graph with no triangles and with minimum degree $\delta \geq 3$. Then, there exist $\delta - 1$ edge disjoint paths of length 8 between any two adjacent vertices in $P_3 G$.*

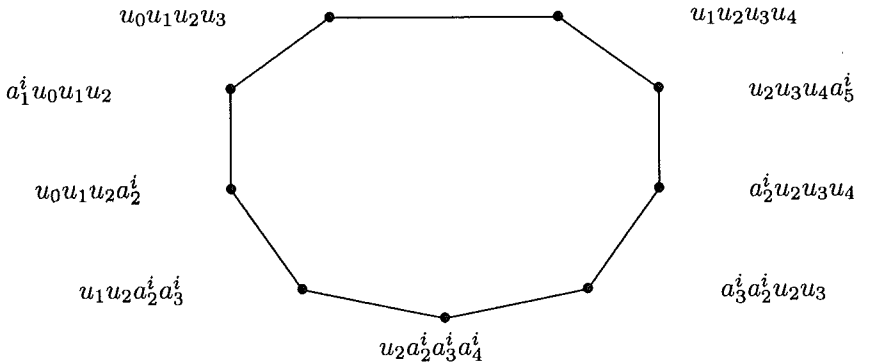
Proof: Let $U = u_0 u_1 u_2 u_3$ and $V = u_1 u_2 u_3 u_4$ be two adjacent vertices in $P_3 G$. Since $\delta \geq 3$, for each $i = 1, \dots, \delta - 1$ there exist vertices

$$\begin{aligned} a_1^i &\in N_G(u_0) - \{u_1\} \\ a_2^i &\in N_G(u_2) - \{u_1\} \\ a_3^i &\in N_G(a_2^i) - \{u_2\} \\ a_4^i &\in N_G(a_3^i) - \{a_2^i\} \\ a_5^i &\in N_G(u_4) - \{u_3\} \end{aligned}$$

If $a_2^i \neq u_3$, since G has no triangles, for each $i = 1, \dots, \delta - 1$ we have a path Q_i ,

$$U, a_1^i u_0 u_1 u_2, u_0 u_1 u_2 a_2^i, u_1 u_2 a_2^i a_3^i, u_2 a_2^i a_3^i a_4^i, \\ u_3 u_2 a_2^i a_3^i, u_4 u_3 u_2 a_2^i, u_2 u_3 u_4 a_5^i, V$$

These paths are edge disjoint because for each value of $i = 1, \dots, \delta - 1$ we can choose different vertices $a_1^i, a_3^i, a_4^i, a_5^i$, so even when choosing a_2^i there are only $\delta - 2$ possibilities to impose $a_2^i \neq u_3$, since $u_i \neq u_j$ if $i \neq j$, the paths can share at most one vertex. ■



At this point we are going to split the study of the connectivity into two cases, depending on the graph G having triangles or not. As it can be seen from the previous lemmas, there are more disjoint paths between adjacent vertices if the graph does not have triangles, or equivalently, the girth is at least 4. Indeed, the minimum degree is larger in path graphs of graphs with no triangles. Notice that a vertex $u_0u_1u_2u_3$ in P_3G is adjacent with the vertices $u_1u_2u_3x$ and $yu_0u_1u_2$, where $x \in N_G(u_3) - \{u_2, u_1\}$ and $y \in N_G(u_0) - \{u_1, u_2\}$, so if $\delta(G) = \delta$, then $\delta(P_3G) \geq 2(\delta - 2)$. However, if G has no triangles it follows immediately that $x \neq u_1$ and $y \neq u_2$, so $\delta(P_3G) = 2(\delta - 1)$.

3.1 P_3 -Path graphs of graphs with triangles

Theorem 3.17 *Let G be a connected graph with minimum degree $\delta \geq 4$, then P_3G is maximally connected.*

Proof: A vertex $U = u_0u_1u_2u_3$ in P_3G is adjacent with the vertices $u_1u_2u_3x$ and $yu_0u_1u_2$, where $x \in N_G(u_3) - \{u_2, u_1\}$ and $y \in N_G(u_0) - \{u_1, u_2\}$, so $\delta(P_3G) = 2(\delta - 2)$ or $\delta(P_3G) = 2(\delta - 2) + 1$. In any case, $\delta(P_3G)$ is determined by the minimum number of vertices in $N_G(u_3) - \{u_2, u_1\}$ and $N_G(u_0) - \{u_1, u_2\}$, taken over all the vertices of P_3G . As it was shown in the proof of Lemma 3.12, every vertex in the form $u_1u_2u_3x$ determines the first edge of a path of length 3 between U and every other vertex adjacent with U . Analogously, every vertex in the form $yu_0u_1u_2$ determines the first edge of a path of length 8 between U and every other vertex adjacent with U . Hence, $\delta(P_3G)$ equals the number of edge disjoint paths joining the endpoints of every edge in G , and we conclude $\delta(P_3G) = \lambda(P_3G)$. ■

Comparing the results in the above theorem to the minimum degree of P_3G we obtain the following corollary.

Corollary 3.18 *Let G be a connected graph with minimum degree $\delta \geq 3$. Then, $\lambda(P_3G) \geq 2(\delta - 2)$. ■*

The study of the superconnectivity makes sense only for maximally connected graphs. For this reason, it is restricted to graphs with $\delta \geq 4$.

Theorem 3.19 *Let G be a connected graph with minimum degree $\delta \geq 5$. Then, P_3G is super- λ .*

Proof: By contradiction, let us suppose that there is a non trivial edge cut in A in P_3G where $|A| = \delta(P_3G)$. From Theorem 3.17, for a given edge $e = \{U, V\}$ in A there exist $\delta(P_3G) - 1$ edge disjoint paths, apart from the edge e , between U and V . Therefore, since A is an edge cut it must

contain at least one edge in each of those paths. Furthermore, since A is non trivial, not all those edges can be adjacent with e . We can assume then that there is an edge, let us say e_i , which is in A and it is not adjacent with e . Let P_i be the path between U and V in which e_i is contained. Since $\delta \geq 5$, and because of Lemma 3.12 and Lemma 3.15, there exists at least one path, let us say P_j , which is disjoint with P_i and have the same length as P_i . Let us suppose,

$$P_i : U = a_0, a_1, \dots, a_l = V$$

$$P_j : U = b_0, b_1, \dots, b_l = V$$

where $l = 3$ or 8 , according to Lemma 3.13 and Lemma 3.16. If $e_i = \{a_r, a_{r+1}\}$, then we can consider the path $U = a_0, a_1, \dots, a_r, b_{r+1}, a_{r+2}, \dots, a_l = V$. If e_i is the only edge of P_i in A , it is not possible for A to be an edge cut. If there are other edges of P_i in A , we can repeat the procedure as many times as needed, in each step eliminating one of those edges from P_i . In any case, the new path constructed is in $G - A$ because it is edge disjoint with the $\delta(P_3G)$ paths joining U and V , and $|A| = \delta(P_3G)$. ■

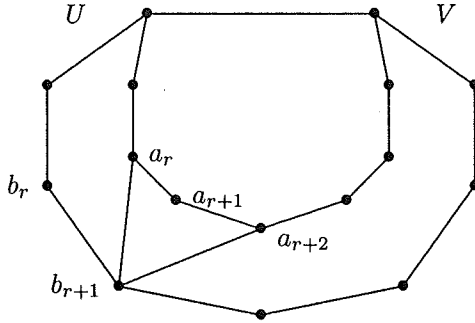


Fig.3 Construction used in Theorem 3.19 and Theorem 3.22 if $l = 8$

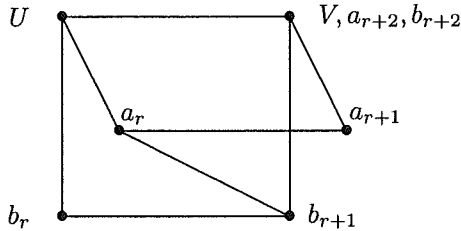


Fig.4 Construction used in Theorem 3.19 and Theorem 3.22 if $l = 3$

3.2 P_3 -Path graphs of graphs with no triangles

For graphs without triangles we can establish an improvement of Theorem 3.17 that can be proved analogously.

Theorem 3.20 *Let G be a connected graph with minimum degree $\delta \geq 3$. Then P_3G is maximally edge connected. ■*

Since G has no triangles, $\delta(P_3G)$ is exactly $2(\delta - 1)$, and this allows us to state the following corollary.

Corollary 3.21 *Let G be a connected graph with no triangles and with minimum degree $\delta \geq 3$. Then, $\lambda(P_3G) = 2(\delta - 1)$. ■*

As a consequence of Theorem 3.20, to study the superconnectivity of graphs without triangles we only need the condition $\delta \geq 3$. Reasoning as in Theorem 3.19 and it can be proved the following result.

Theorem 3.22 *Let G be a connected graph with no triangles and with minimum degree $\delta \geq 4$. Then, P_3G is super- λ . ■*

4 Iterated P_3 path graphs

For a simple graph G , the graph P_3^nG is defined as $P_3^nG = P_3G$ if $n = 1$, and $P_3^nG = P_3(P_3^{n-1}G)$, otherwise. In order to extend the results about connectivity and superconnectivity of P_3G to any P_3^nG , the foundations are the conditions on G to guarantee that P_3^nG is connected that were established in Corollary 3.9 and Corollary 3.11.

By induction on n and using Theorem 3.17 the next result can be obtained.

Theorem 4.1 *Let G be a connected graph with minimum degree $\delta \geq 4$. Then P_3^nG is maximally connected. ■*

Again by induction on n and applying Corollary 3.18 the following corollary can be shown.

Corollary 4.2 *Let G be a connected graph with minimum degree $\delta \geq 4$. Then, $\lambda(P_3^nG) \geq 2^n\delta - 2^{n+2} + 4$. ■*

In the case of graphs without triangles, as a consequence of Corollary 3.14, we can respectively apply Theorem 3.20 and Corollary 3.21 to improve the two previous results.

Theorem 4.3 *Let G be a connected graph with no triangles and with minimum degree $\delta \geq 3$. Then $P_3^n G$ is maximally edge connected. ■*

Corollary 4.4 *Let G be a connected graph with no triangles and with minimum degree $\delta \geq 3$. Then, $\lambda(P_3^n G) = 2^n \delta - 2^{n+1} + 2$. ■*

For maximally connected $P_3^n G$ path graphs it is interesting to study the superconnectivity. The next two results follow from Theorem 3.19 and Theorem 3.22 by induction on n .

Theorem 4.5 *Let G be a connected graph with minimum degree $\delta \geq 5$. Then $P_3^n G$ is super- λ . ■*

Theorem 4.6 *Let G be a graph connected graph with no triangles and with minimum degree $\delta \geq 4$. Then, $P_3^n G$ is super- λ . ■*

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