# A technique for the $(d, s, N)$-bus network problem

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## Abstract

*A problem in the design of bus interconnection networks is to find directed hypergraphs with minimum diameter for fixed values of the order, processor degree and bus size. In this paper we propose the partial line hyperdigraph as a technique for it. The partial line hyperdigraph is related to the line hyperdigraph [1], the partial line digraph [6] and the line digraph [7]. Partial line hyperdigraphs have also good connectivity, expandability and easy routing. Specially interesting results are obtained for the generalized Kautz hyperdigraphs.*

## 1 Introduction

Hypergraphs are a useful generalization of graphs for Computer Science and Discrete Mathematics [8]. We focus on its application to designing bus networks. That is, communication systems made up of buses communicating several processors. Bus networks have better reliability and performance that those based on point-to-point connections. They are modeled by hypergraphs (bidirectional buses) or directed hypergraphs (unidirectional buses). We deal with directed hypergraphs, also called hyperdigraphs.

A processor can only be connected to a limited number of buses. A bus can only accept a given number of connections to processors. Then, to communicate some nodes it is necessary to traverse intermediate nodes, and the transmission delay depends on such number of nodes. Thus, a problem is to connect by buses an arbitrary number of processors minimizing the transmission delay. This gives rise to the need of hyperdigraphs with arbitrary order, maximum processor degree, maximum bus size and minimum diameter. A powerful technique for this problem is the line hyperdigraph proposed in [1]. As particular solutions, in [3] were introduced the generalized De Bruijn and generalized Kautz hyperdigraphs. They are iterated line hyperdigraphs. In this work we present a generalization of the line hyperdigraph technique which makes it more versatile.

Some of the basic definitions and notation for the following are in the next Section. The definition of the partial line hyperdigraph and its relation with some other known techniques involving hyperdigraphs and digraphs is presented in Section 3. In Section 4 is shown its usefulness to the design of bus interconnection networks. The application to Kautz hyperdigraphs is in Section 5.

## 2 Preliminaries

A *directed hypergraph*, or *hyperdigraph* $H$ is a pair $(\mathcal{V}(H), \mathcal{E}(H))$, where $\mathcal{V}(H)$ is a nonempty set of *vertices*, and $\mathcal{E}(H)$ is a set of ordered pairs of non-empty subsets of $\mathcal{V}(H)$, called *buses*. If $E=(E^-,E^+)$ is a bus, we say that $E^-$ is the *in-set*, $E^+$ is the *out-set* of $E$, and that $E$ joins vertices in $E^-$ to vertices in $E^+$. Its *in-size*(*out-size*) is the cardinal of $E^-$, $|E^-|(|E^+|)$. If $v$ be a vertex, the *in-degree*(*out-degree*) of $v$ is the number of buses containing $v$ in the out-set(in-set), and is denoted by $d^-(v)(d^+(v))$.

If $H$ is a hyperdigraph, its *order* is the number of vertices, $|\mathcal{V}(H)|$, denoted by $n(H)$, and $m(H)$ will be the number of buses. The *maximum in-size* and *maximum out-size* of $H$ are respectively defined by

$$s^-(H)=\max\{|E^-|:E\in\mathcal{E}(H)\}\ ,\\ s^+(H)=\max\{|E^+|:E\in\mathcal{E}(H)\}$$

Similarly, the *maximum in-degree* , *maximum out-degree* of $H$ are

$$d^-(H)=\max\{d^-(v):v\in\mathcal{V}(H)\}\ ,\\ d^+(H)=\max\{d^+(v):v\in\mathcal{V}(H)\}$$

We denote $s(H)=\max\{s^+(H),s^-(H)\}$ , $d(H)=\max\{d^+(H),d^-(H)\}$.

A *path* of *length* $k$ from a vertex $u$ to a vertex $v$ in $H$ is an alternating sequence of vertices and buses $u=v_0,E_1,v_1,E_2,v_2,\ldots,E_k,v_k=v$ such that $v_i\in E^-_{i+1}$, $(i=0,\ldots,k-1)$ and $v_i\in E^+_i$, $(i=1,\ldots,k)$. The *distance* from $u$ to $v$ is the length of the shortest path from $u$ to $v$. The *diameter* of $H$, $D(H)$, is the maximum distance between every pair of vertices of $H$.

A hyperdigraph is *connected* if there exists at least one path from each vertex to any other vertex. The *vertex-connectivity*, $\kappa(H)$, of a hyperdigraph $H$, is the minimum number of vertices to be removed to obtain a non-connected or trivial hyperdigraph (a hyperdigraph with only one vertex). Similarly is defined the *bus-connectivity*, $\lambda(H)$.

The *dual hyperdigraph*, $H^*$, of a hyperdigraph $H$ has its set of vertices in one-to-one correspondence with the set of buses of $H$, and for every vertex $v$ of $H$ it has a bus, $(V^-,V^+)$, such that a vertex $e\in V^-$ if and only if $v\in E^+$ and $e\in V^+$ if and only if $v\in E^-$.

The *underlying digraph* of a hyperdigraph $H$ is the digraph $\widehat{H}=(\mathcal{V}(\widehat{H}),\mathcal{A}(\widehat{H}))$ with $\mathcal{V}(\widehat{H})=\mathcal{V}(H)$ and $\mathcal{A}(\widehat{H})=\{(u,v):\exists E\in\mathcal{E}(H),u\in E^-,v\in E^+\}$. That is, there is an arc from a vertex $u$ to a vertex $v$ in $\widehat{H}$ if and only if there is an bus joining $u$ to $v$ in $H$. So, paths in $\widehat{H}$ and $H$ are in correspondence, and this implies $D(\widehat{H})=D(H)$ and $\kappa(\widehat{H})=\kappa(H)$.

The *line hyperdigraph* of $H$ [1] is the hyperdigraph $LH=(\mathcal{V}(LH),\mathcal{E}(LH))$,

$$\mathcal{V}(LH)=\cup_{E\in\mathcal{E}(H)}\{(uEv):u\in E^-,v\in E^+\}\\ \mathcal{E}(LH)=\cup_{v\in\mathcal{V}(H)}\{(EvF):v\in E^+,v\in F^-\}$$

with $(EvF)^-=\{(wEv):w\in E^-\}$ and $(EvF)^+=\{(vFw):w\in F^+\}$. Note that if $H$ is a digraph, $LH$ coincides with the line digraph of $H$ [7]. We refer [1] for other properties of the line hyperdigraph technique.

# 3 The technique

Given a hyperdigraph $H=(\mathcal{V}(H),\mathcal{E}(H))$ with minimum in-degree at least 1 for any set $\mathcal{V}'$ of vertices of $LH$ such that $\{v:\exists(uEv)\in\mathcal{V}'\}=\mathcal{V}(H)$, the partial line hyperdigraph of $H$ will be the hyperdigraph $\mathcal{L}H=(\mathcal{V}(\mathcal{L}H),\mathcal{E}(\mathcal{L}H))$,

$$\begin{aligned}&\mathcal{V}(\mathcal{L}H)=\mathcal{V}'\\ &\mathcal{E}(\mathcal{L}H)=\{(EvF):v\in F^-,\exists u\in\mathcal{V}(H):\\ &\qquad\qquad\qquad (uEv)\in\mathcal{V}'\}\\ &(EvF)^+=\{(vFw):(vFw)\in\mathcal{V}'\}\cup\\ &\qquad\qquad \{(v'F'w):w\in F^+,(vFw)\notin\mathcal{V}'\}\\ &(EvF)^-=\{(uEv):u\in E^-,(uEv)\in\mathcal{V}'\}\end{aligned}$$

That is, $(EvF)^+$ contain all the vertices in the form $(vFw)$ of $\mathcal{V}'$, and one arbitrary vertex, $(v'F'w)$, if $(vFw)$ is not in $\mathcal{V}'$.

That is, the partial line hyperdigraphs

depends on the choose of $\mathcal{V}'$ and also in the way that the out-sets of the buses are constructed.

Note that always exists a set $\mathcal{V}'$ with $\{v : (uEv) \in \mathcal{V}'\} = \mathcal{V}(H)$, because the minimum degree of $H$ is at least 1.

Particularly, observe that in the case $|\mathcal{V}'| \geq ds$, we can take the out-set of the buses in such a way that $\mathcal{E}(\mathcal{L}H) = \mathcal{E}(LH)$. In fact, for any vertex $v$ and any bus $E$ of $H$ such that $v \in E^+$ and $(uEv) \in \mathcal{V}'$, it is possible to take $(EvF)^+ = \{(vFw) : (vFw) \in \mathcal{V}'\} \cup \{(v'Fw) : w \in F^+, (vFw) \notin \mathcal{V}'\}$.

Notice that if $H$ is a digraph, $\mathcal{L}H$ coincides with a partial line digraph. Also, if $\mathcal{V}' = \mathcal{V}(LH)$ then $\mathcal{L}H$ is $LH$. So, the partial line hyperdigraph technique is a generalization of the line hyperdigraph technique [1], the partial line digraph [6], and consequently, the line digraph [7].

Next, we show some useful relations of this technique to digraphs.

**Proposition 3.1** *Let $H = (\mathcal{V}(H), \mathcal{E}(H))$ with minimum $d > 1$, and $\mathcal{V}'$ a set of vertices of $LH$ such that $\{v : \exists (uEv) \in \mathcal{V}'\} = \mathcal{V}(H)$. For any vertex $(uEv)$ and any bus $(EvF)$ of the partial line hyperdigraph of $H$, $\mathcal{L}H$,*

$d^+_{\mathcal{L}H}(uEv) = d^+_H(v)$;
$s^+_{\mathcal{L}H}(EvF) = s^+_H(F)$. □

**Proposition 3.2** *Let $H$ be a hyperdigraph with minimum in-degree $d > 1$. There exists a set of vertices of $LH$, and a set of arcs of $\widehat{H}$, such that with these sets $\widehat{\mathcal{L}H}$ and $\mathcal{L}\widehat{H}$ are isomorphic.*

*Proof*: Given a set $\mathcal{V}'$ of vertices of $LH$ such that $\{v : (uEv) \in \mathcal{V}'\} = \mathcal{V}(H)$. Let $E'$ be the set of arcs of $\widehat{H}$ defined by $E' = \{(u, v) : \exists E \in \mathcal{E}(H), (uEv) \in \mathcal{V}'\}$. With these sets there is a trivial isomorphism between $\widehat{\mathcal{L}H}$ and $\mathcal{L}\widehat{H}$. □

# 4 The $(d, s, N)$-problem

In [4] there is a Moore like bound for the order of a hyperdigraph of diameter $D$, maximum out-degree $d$ and maximum out-size $s$:
$N \leq 1 + (ds) + (ds)^2 + \ldots + (ds)^D = \frac{(ds)^{D+1} - 1}{ds - 1}$.
From this arises the following lower bound for the diameter:

$(\log_{ds}(N(ds - 1) + 1)) - 1 \leq D$.

We will show the good behavior of the proposed technique for such problem.

The order of the hyperdigraph $\mathcal{L}H$ is the cardinal of $\mathcal{V}'$, and it is chosen with the condition $\{v : (uEv) \in \mathcal{V}'\} = \mathcal{V}(H)$. Then,

$$|\mathcal{V}(H)| \leq |\mathcal{V}(\mathcal{L}H)| \leq |\mathcal{V}(LH)|.$$

The partial line hyperdigraph preserves the maximum out-degree of $H$. In fact, $d^+_{\mathcal{L}H}(uEv) = d^+_H(v)$ for any vertex $(uEv)$ of $\mathcal{L}H$. Also, the out-size of $H$ remains constant since for every bus $(EvF)$ of $\mathcal{L}H$, $s^+_{\mathcal{L}H}(EvF) = s^+_H(F)$ (the in-size of $H$ is preserved too). Then, if $H$ is $d$-regular, $\mathcal{L}H$ is also $d$-regular. So, if the out-size of all buses of $H$ is $s$,

$$|\mathcal{V}(H)| \leq |\mathcal{V}(\mathcal{L}H)| \leq |\mathcal{V}(H)|ds$$

Since $D(H) = D(\widehat{H})$ for every hyperdigraph $H$, by Proposition 3.2 we have $D(\mathcal{L}H) = D(\widehat{\mathcal{L}H})$. Now, $\widehat{H}$ is a digraph and by [6]: $D(\widehat{H}) \leq D(\mathcal{L}\widehat{H}) \leq D(\widehat{H}) + 1$. So, $D(H) \leq D(\mathcal{L}H) \leq D(H) + 1 = D(LH)$.

From all the above considerations about the order, maximum out-degree and maximum out-size, we can state the following result:

**Theorem 4.1** *Let $H$ be a hyperdigraph with maximum out-degree $d > 1$, maximum out-size $s$, order $N$ and diameter $D$. Then the order $N_{\mathcal{L}}$, the maximum out-degree $d_{\mathcal{L}}$, the maximum out-size $s_{\mathcal{L}}$ and the diameter $D_{\mathcal{L}}$ of any partial line hyperdigraph $\mathcal{L}H$ satisfy:*

$$N \leq N_{\mathcal{L}} \leq Nds; \qquad d_{\mathcal{L}} = d;$$
$$D \leq D_{\mathcal{L}} \leq D + 1. \qquad s_{\mathcal{L}} = s.\square$$

### 4.1 Connectivity

To show that the partial line hyperdigraph tends to increase the connectivity (with the minimum degree of the partial line hyperdigraph as a lower bound), first, we extend a useful concept introduced in [6] for digraphs. A hyperdigraph $H$ has *no redundant short paths* when there is at most one path of length one or two between every pair of vertices (different or not) of $H$. Notice that under this restriction we can still work with interesting hyperdigraphs. For instance, the generalized De Bruijn hyperdigraphs and the generalized Kautz hyperdigraphs [3] have no redundant short paths.

**Lemma 4.2** *Let $H$ be a hyperdigraph. Then, $H$ has no redundant short paths if and only if $\widehat{H}$ has no redundant short paths.* $\square$

**Theorem 4.3** *Let $H$ be a hyperdigraph with minimum in-degree $d > 1$ and minimum in-size $s$. If $H$ has no redundant short paths:*

$$\min\{\kappa(H), d(\mathcal{L}H)s\} \leq \kappa(\mathcal{L}H)$$

*Proof*: By Lemma 4.2, $\widehat{H}$ has no redundant short paths, so by the bound on the connectivity of partial line digraphs [6], $\min\{\kappa(\widehat{H}), d(\mathcal{L}\widehat{H})\} \leq \kappa(\mathcal{L}\widehat{H})$. Since $\kappa(H) = \kappa(\widehat{H})$, then $\min\{\kappa(H), d(\mathcal{L}H)s\} \leq \kappa(\mathcal{L}H)$. $\square$

For the bus-connectivity the analogous bound holds, but to prove it, we need the following result of [1]:

**Lemma 4.4** *Let $H$ be a hyperdigraph with bus-connectivity $\lambda$. Then, every vertex $v$ in $H$ is on $\lambda$ bus-disjoint cycles.*$\square$

**Theorem 4.5** *Let $H$ be a hyperdigraph with minimum in-degree $d > 1$. Let $\mathcal{V}'$ be a set of vertices of $LH$, $|\mathcal{V}'| \geq ds$, and $\mathcal{L}H$ a partial line hyperdigraph with $\mathcal{E}(\mathcal{L}H) = \mathcal{E}(LH)$. If $H$ has no redundant short paths:*

$$\min\{\lambda(H), d(\mathcal{L}H)\} \leq \lambda(\mathcal{L}H)$$

*Proof*: If $\min\{\lambda(H), d(\mathcal{L}H)\} = d(\mathcal{L}H)$, let us see that any set $F$ of buses of $\mathcal{L}'H$ with $|F| < d(\mathcal{L}'H)$ cannot disconnect $\mathcal{L}H$. In fact, for any path between two given vertices of $\mathcal{L}H$, let us say, $(uEv)$ and $(xFy)$, a path $\mathcal{L}P$ in $\mathcal{L}H$ is in correspondence with a path $P$ between $v$ and $x$ in $H$ in the following way:

$$\begin{aligned}\mathcal{L}P = (uEv), (EvE'_1), (v'E'_1v_1), (E'_1v_1E'_2),\\ \dots (v'_nE'_nx), (E'_nx'F), (xFy)\\ P = v, E_1, v_1, E_2, v_2, \dots, E_{n-1}, v_{n-1}, E_n, x\end{aligned}$$

Now, if $F = \{(E_iv_iF_i) : i = 1, \dots, r\}$, $0 < r < d(\mathcal{V}'H)$, is a disconnecting set of $\mathcal{L}H$, the set $F' = \{F_i : i = 1, \dots, r\}$ is a disconnecting set of $H$, which contradicts that $\min\{\lambda(H), d(\mathcal{L}H)\} = d(\mathcal{L}H)$.

If not, let see that $\lambda(H) \leq \lambda(\mathcal{L}H)$. For this, is enough to show that a set of $\lambda$ bus-disjoint paths in $H$ induces a set of $\lambda$ bus-disjoint paths in $\mathcal{L}H$, if $\lambda \leq \min\{\lambda(H), d(\mathcal{L}H)\}$. Let $(uEv)$ and $(xFy)$ be two different vertices of $\mathcal{L}H$. To construct $\lambda$ bus-disjoint paths from $(uEv)$ to $(xFy)$ in $\mathcal{L}H$ from $\lambda$ bus-disjoint paths from $v$ to $x$ in $H$, we consider two cases:

1. If $v \neq x$, we have $\lambda$ paths from $v$ to $x$: $P_i = v, E^i_1, v^i_1, E^i_2, v^i_2, \dots, E^i_{n_i-1}, v^i_{n_i-1}, E^i_{n_i}, x$ with $1 \leq i \leq \lambda$. Each path $P_i$ give rise to a path from $(uEv)$ to $(xFy)$, $\mathcal{L}P_i$ in $\mathcal{L}H$ defined by: $\mathcal{L}P_i = (uEv), (EvE'^i_1), (v'E'^i_1v^i_1), (E'i_1v^i_1E'^i_2), \dots (E'^i_{n_i}x'F), (xFy)$.
   It is possible to construct $\mathcal{L}P_i$ because for every vertex of $H$ there exists at least one bus arriving to it. These paths are bus-disjoint because the paths in $H$ are

also, and because of the restriction to hyperdigraphs with no redundant short paths. Besides, these paths are vertex-disjoint, because if two different paths $\mathcal{L}P_i$, $\mathcal{L}P_j$, have a common vertex, let us say, $({v'}^i_{r+1}E^i_{r+1}v^i_{r+1}) = ({v'}^j_{s+1}E^j_{s+1}v^j_{s+1})$, then should exist in $H$ two different paths with length 1 or 2 from ${v'}^i_{r+1} = {v'}^j_{s+1}$ to $v^i_{r+1} = v^j_{s+1}$, contradicting the no redundant short paths restriction.

2. If $v = x$, we proceed as before but with bus disjoint cycles in $H$. By Lemma 4.4, if the bus-connectivity is $\lambda$, each vertex of $H$ is in $\lambda$ bus-disjoint cycles. In the same way as we do with paths $P_i$, we can obtain $\lambda$ cycles in $\mathcal{L}H$. Again, since the original cycles are bus-disjoint and $H$ has no redundant short paths, these new cycles are bus-disjoint also. □

## 4.2 Expandability

Given two hyperdigraphs $H$ and $H'$, on $N$ and $N'$ vertices, respectively, $N \leq N'$, we define the *index of expandability* of $H$ to $H'$, $e(H, H')$, as the minimum number of buses that has to be modified or removed from $H$ to obtain $H'$ by adding $N' - N$ vertices and some appropriate buses, if it is necessary.

That is, the index of expandability measures the necessary modifications of buses of $H$, to obtain a sub-hyperdigraph $H'$.

Let $\mathcal{L}_nH$ be a partial line hyperdigraph of $H$ with order $n$. Next we show that any $\mathcal{L}_nH$ has good expandability to some $\mathcal{L}_{n+1}H$.

**Theorem 4.6** *Let $H = (\mathcal{V}(H), \mathcal{E}(H))$ be a hyperdigraph with maximum in-degree $d > 1$. For any partial line hyperdigraph $\mathcal{L}_nH$ on $n$ vertices, $|\mathcal{V}(H)| \leq n \leq |\mathcal{V}(LH)| - 1$, there exists a digraph $\mathcal{L}_{n+1}H$, such that the index of expandability of $\mathcal{L}_nH$ to $\mathcal{L}_{n+1}H$ satisfies:*

$$e(\mathcal{L}_nG, \mathcal{L}_{n+1}G) \leq 2d$$

*Proof*: Let $\mathcal{V}'$ be the set of vertices of $\mathcal{L}_nH$. The hyperdigraph $\mathcal{L}_{n+1}H$ can be obtained from $\mathcal{L}_nH$ by the following algorithm:

*a)* Choose a vertex $(uEv)$ of $\mathcal{L}H$ out of $\mathcal{V}'$. Since $|\mathcal{V}'| \leq |\mathcal{V}(LH)| - 1'$, it exists.
*b)* Add the vertex $(uEv)$ to $\mathcal{L}H$.
*c)* For every bus of $\mathcal{L}H$, $(FuE)$, replace in their out-sets, the vertex $(u'E'v)$ by $(uEv)$.
*d)* For every bus $F$ of $H$, if $(EvF)$ is not a bus of $\mathcal{L}H$, add it, with
$(EvF)^+ = \{(vFw) : (vFw) \in \mathcal{V}'\} \cup$
$\{(v'F'w) : w \in F^+, (vFw) \notin \mathcal{V}'\}$.
$(EvF)^- = \{(uEv) : u \in E^-, (uEv) \in \mathcal{V}'\}$.
For each $F$ such that $(EvF)$ is a bus of $\mathcal{L}H$, put the vertex $(uEv)$ in the in-set.
We only add new buses or replace the existing ones in steps c) and d), so the index of expand ability is given by the number of changes there. Since the maximum degree is $d$, this number is at most $2d$. □

The above proof gives an algorithm to expand partial line hyperdigraphs. With a few changes it can be used to decrease the number of vertices.

Also in some applications, it could also be useful to measure the number of vertex-to-vertex connections that have to be modified to add components. From the above algorithm:

**Corollary 4.7** *Let $H = (\mathcal{V}(H), \mathcal{E}(H))$ be a hyperdigraph with maximum in-degree $d > 1$ and maximum out-size $s$. For any partial line hyperdigraph $\mathcal{L}_nH$ on $n$ vertices, $|\mathcal{V}(H)| \leq n \leq |\mathcal{V}(LH)| - 1$, there exists a hyperdigraph $\mathcal{L}_{n+1}H$, such that the connections that have to be modified to transform $\mathcal{L}_nH$ to $\mathcal{L}_{n+1}H$ are at most $ds$.* □

# 5 Applications

Let us see the application of the partial line hyperdigraph technique to the generalized Kautz hyperdigraphs, introduced in [3]. There it was proved that $LGK(d,n,s,m) = GK(d,dsn,s,dsm)$ and $\widehat{GK}(d,n,s,m)$ is the generalized Kautz digraph, $GK(ds,n)$ [10, 11].

**Theorem 5.1** *For any positive integers $d,n,s,m$, with $dn \equiv_m 0$, $sm \equiv_n 0$.*
*If $D(GK(d,n,s,m)) > 4$, then:*

$\kappa(GK(d,n,s,m)) = ds$,*$n$ multiple of $(ds+1)$ and $gcd(n,ds) > 1$;*
$\kappa(GK(d,n,s,m)) = ds-1$, *otherwise.*

*Proof*: $\kappa(GK(d,n,s,m)) = \kappa(\widehat{GK}(d,n,s,m))$ which coincides with $\kappa(GK(ds,n))$. From [9] if $D(GK(ds,n)) > 4$, it has maximum vertex-connectivity $(ds)$ when $n$ is a multiple of $(ds+1)$ and $gcd(n,ds) > 1$, and $ds-1$ otherwise. Since $D(GK(d,n,s,m)) = D(\widehat{GK}(d,n,s,m)) = D(GK(ds,n))$, we have the result. □

**Corollary 5.2** *For any positive integers $d,n,s,m$, with $dn \equiv_m 0$, $sm \equiv_n 0$.*
*Let $H$ be $GK(d,n,s,m)$ and $D(\mathcal{L}H) > 4$:*

$\kappa(\mathcal{L}H) \geq d(\mathcal{L}H)s$,*$n$ multiple of $(ds+1)$ and $gcd(n,ds) > 1$;*
$\kappa(\mathcal{L}H) \geq \min\{ds-1, d(\mathcal{L}H)s\}$, *otherwise.*

*Proof*: There are not redundant paths, so by Theorem 4.3: $\kappa(\mathcal{L}GK(d,n,s,m))$ is at least $\min\{\kappa(GK(d,n,s,m)), d(\mathcal{L}GK(d,n,s,m))s\}$.It remains to apply the above theorem. □

Since $\widehat{H}$ is a digraph, $\kappa(\widehat{H}) \leq \lambda(\widehat{H})$ [5], and it is easy to see that $\lambda(\widehat{H}) \leq \lambda(H)s$, similar results are valid for the bus-connectivity.

**Theorem 5.3** *For any positive integers $d,n,s,m$ with $dn \equiv_m 0$, $sm \equiv_n 0$.*
*If $D(GK(d,n,s,m)) > 4$, then:*

$\lambda(GK(d,n,s,m)) = d$,*$n$ multiple of $(ds+1)$ and $gcd(n,ds) > 1$;*
$\lambda(GK(d,n,s,m)) = d$ *or* $d-1$, *otherwise.*

**Corollary 5.4** *For any positive integers $d,n,s,m$, with $dn \equiv_m 0$, $sm \equiv_n 0$.*
*Let $H$ be $GK(d,n,s,m)$ and $D(\mathcal{L}H) > 4$:*

$\lambda(\mathcal{L}H) \geq d(\mathcal{L}H)$,*$n$ multiple of $(ds+1)$ and $gcd(n,ds) > 1$;*
$\lambda(\mathcal{L}H) \geq \min\{d-1, d(\mathcal{L}H)\}$, *otherwise.*□

We have studied the connectivity of the partial line hyperdigraphs of the generalized Kautz hyperdigraphs. Now, let us see other interesting properties of such hyperdigraphs.

**Theorem 5.5** *Let $H$ be a hyperdigraph with minimum in-degree $d > 1$. There exists a set of vertices of $LH$, and a set of vertices of $L^2H$, such that with these sets it is possible to construct $L\mathcal{L}H$ and $\mathcal{L}LH$ to be isomorphic.*

*Proof*: The vertices of $L\mathcal{L}H$ are in correspondence with the buses of $\mathcal{L}H$, so there are two kinds of vertices:

1. $(uEv)(EvF)(v'F'w)$, with $v \in F^-$, $w \in F^+$ and $(vFw) \notin V(\mathcal{L}H)$
2. $(uEv)(EvF)(vFw)$, with $v \in F^-$, $w \in F^+$ and $(vFw) \in V(\mathcal{L}H)$

Clearly, for any choice of vertices of $LH$, there are different digraphs $\mathcal{L}H$ and $L\mathcal{L}H$. For a given digraph $L\mathcal{L}H$, we construct a set of vertices of $LH$ by the rules:

1. If $(uEv)(EvF)(v'F'w) \in V(L\mathcal{L}H)$, we take $(uEv)(EvF)(vFw)$
2. If $(uEv)(EvF)(vFw) \in V(L\mathcal{L}H)$, we take $(uEv)(EvF)(vFw)$

Applying the partial line technique to $LH$ with this set of vertices, it is possible to construct $L\mathcal{L}H$ and $\mathcal{L}LH$ to be isomorphic. □

**Corollary 5.6** *Let $H$ be a hyperdigraph with minimum in-degree $d > 1$. There exists a set of vertices of $LH$, and a set of vertices of $L^{k+1}H$, such that with these sets, for any* *integer $k \geq 1$, it is possible to construct $L^k\mathcal{L}H$ and $\mathcal{L}L^kH$ to be isomorphic.*

*Proof*: By the above theorem, there exists a set of vertices of $LH$, and a set of vertices of $H$, such that $L\mathcal{L}H$ and $\mathcal{L}LH$. (The result for $k = 1$.) We are going to prove by induction on $k$ that the same holds for any $k \geq 1$. Let us assume that $L^i\mathcal{L}H$ and $\mathcal{L}L^iH$ are isomorphic for any integer $i$, $1 \leq i \leq k-1$, and we are going to prove $L^k\mathcal{L}H$ and $\mathcal{L}L^kH$ are so:
$\mathcal{L}L^kH = \mathcal{L}L^{k-1}LH \sim L^{k-1}\mathcal{L}LH \sim L^l\mathcal{L}H$. $\square$

$L^kGK(d,n,s,m) = GK(d,(ds)^kn,s,(ds)^km)$, for any positive integer $k$ [1]. Now, for an integer, $\alpha$, $1 < \alpha < ds$, notice that a partial line hyperdigraph on $\alpha n(ds)^k$ vertices, $\mathcal{L}GK(d,(ds)^kn,s,(ds)^km)$ holds:

$$\mathcal{L}GK(d,(ds)^kn,s,(ds)^km) = \mathcal{L}L^kGK(d,n,s,m) = L^k\mathcal{L}GK(d,n,s,m).$$

The number of buses of $\mathcal{L}GK(d,n,s,m)$ is the number of paths of length 1 in its dual [1], $\alpha n$. In Section 4 it was proved:

1. $d^+_{\mathcal{L}H}(uEv) = d^+_H(v)\ \forall(uEv)$ vertex of $\mathcal{L}H$.
2. $s^+_{\mathcal{L}H}(EvF) = s^+_H(F)\ \forall(EvF)$ bus of $\mathcal{L}H$.

So, some digrahs with the same properties for the $(d,n)$-problem that some Imase-Itoh or generalized Kautz digraphs can be obtained as partial line digraphs of Kautz digraphs.(We recall that the $(d,n)$-digraph problem is the same that the $(d,n,1)$-hyperdigraph problem, since for $s = 1$, hyperdigraphs are digraphs). Particularly, if we let $n$ to be a multiple of $ds+1$, we can obtain those of maximum connectivity by the application of the corresponding partial line operator.

# References


[1] J.C. Bermond, F. Ergincan. Directed Line Hypergraphs,*Private Communication.*

[2] J.C. Bermond, F. Ergincan. Bus interconnection networks, *Discrete Appl.Math.*,**68** (1996) 1–15.

[3] J.C. Bermond, R.W. Dawes, F. Ergincan. De Bruijn and Kautz Bus Networks, *Networks*,**30** (1997) 205–218.

[4] F. Ergincan, D.A. Gregory. Directed Moore Hypergraphs, Technical Report 1992-08, Queens University at Kingston, Dept. of Math. and Stats..(1992). *Discrete Appl.Math.* **63** Springer (1995) 117–127.

[5] J. Fàbrega, M.A. Fiol. Maximally Connected Digraphs, *Journal of Graph Theory* **13-6** (1999), 657–688.

[6] M.A. Fiol, A.S. Lladó. The Partial Line Digraph Technique in the Design of Large Interconnection Networks, *IEEE Trans. on Comp.* **41/7** (1992), 848–857.

[7] M.A. Fiol, J.L.A. Yebra, I. Alegre. Line-digraph Iterations and the $(d,k)$ Problem, *IEEE Trans. on Comp.* **C-33** (1984), 400–403.

[8] G. Gallo, G. Longo, S. Pallotino. Directed hypergraphs and applications, *Discrete Applied Mathematics* **42** (1993) 177–201.

[9] N. Homobono, C. Peyrat. Connectivity of Imase and Itoh digraphs, *IEEE Trans. on Comp.* **C-37** (1988), 1459–1461.

[10] M. Imase, M. Itoh. Design to Minimize Diameter on Building-Block Network, *IEEE Trans. on Comp.* **C-30** (1981), 439–442.

[11] M. Imase, M. Itoh. A Design for Directed Graphs with Minimum Diameter, *IEEE Trans. on Comp.* **C-32** (1983), 782–784.