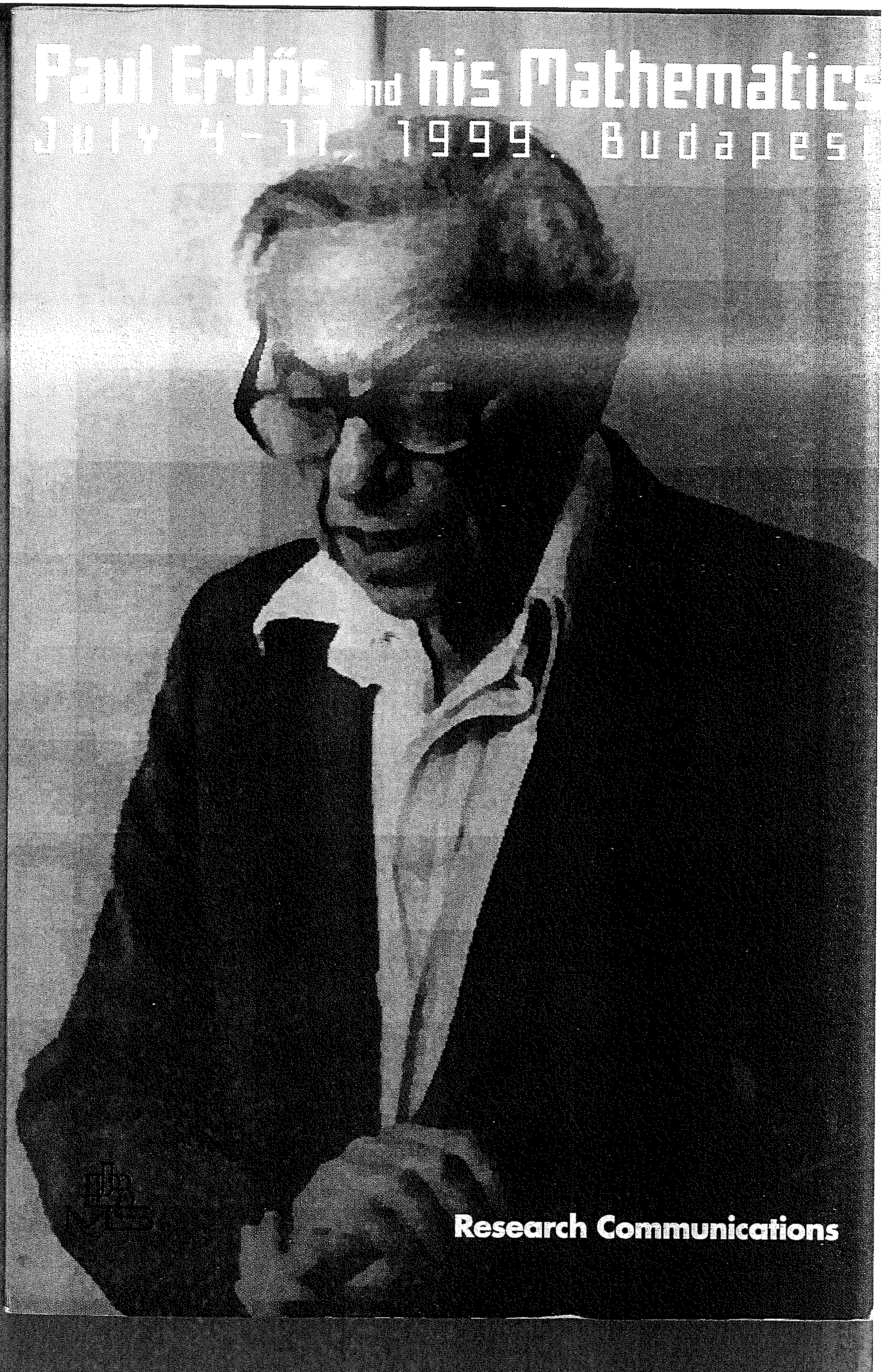
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Maximally connected hyperdigraphs

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Extended abstract

A directed hypergraph, or simply hyperdigraph, H is a pair $(\mathcal{V}(H), \mathcal{E}(H))$, where $\mathcal{V}(H)$ is a non-empty set of *vertices*, and $\mathcal{E}(H)$ is a set of ordered pairs of non-empty subsets of $\mathcal{V}(H)$, called *hyperarcs*. If $E = (E^-, E^+)$ is a hyperarc, we say that E^- is the *in-set*, E^+ is the *out-set* of E, and that E joins vertices in E^- to vertices in E^+. Its *in-size*(*out-size*) is the cardinal of E^-, $|E^-|(|E^+|)$. If v is a vertex, the *in-degree*(*out-degree*) of v is the number of hyperarcs containing v in the out-set(in-set), and it is denoted by $d^-(v)(d^+(v))$. Its *order* is the number of vertices, $|\mathcal{V}(H)|$, denoted by $n(H)$, and its *size*, is the number of hyperarcs, $m(H)$. The *maximum in-size* and *maximum out-size* of H are respectively defined by

$$s^-(H) = \max\{|E^-| : E \in \mathcal{E}(H)\},\ s^+(H) = \max\{|E^+| : E \in \mathcal{E}(H)\}$$

Similarly, the *maximum in-degree*, *maximum out-degree* of H are

$$d^-(H) = \max\{d^-(v) : v \in \mathcal{V}(H)\},\ d^+(H) = \max\{d^+(v) : v \in \mathcal{V}(H)\}$$

We denote $s(H) = \max\{s^+(H), s^-(H)\}$, $d(H) = \max\{d^+(H), d^-(H)\}$. Note that when $s = 1$, H is a digraph.

The *underlying digraph* of a hyperdigraph H is $\widehat{H} = (\mathcal{V}(\widehat{H}), \mathcal{A}(\widehat{H}))$ with $\mathcal{V}(\widehat{H}) = \mathcal{V}(H)$ and $\mathcal{A}(\widehat{H}) = \{(u,v) : \exists E \in \mathcal{E}(H), u \in E^-, v \in E^+\}$.

A hyperdigraph is *connected* if there exists at least one path from each vertex to any other vertex. The *vertex-connectivity*, $\kappa(H)$, of a hyperdigraph H, is the minimum number of vertices to be removed to obtain a non-connected or trivial hyperdigraph (a hyperdigraph with only one vertex). Similarly is defined the *hyperarc-connectivity*, $\lambda(H)$.

The *line hyperdigraph* of $H = (\mathcal{V}(H), \mathcal{E}(H))$ is defined in [1] as the hyperdigraph $LH = (\mathcal{V}(LH), \mathcal{E}(LH))$,

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digraphs

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$E^+|$). If v is a vertex, the
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rcs, $m(H)$. The *maximum*
ned by

$\{|E^+| : E \in \mathcal{E}(H)\}$

of H are

$\{d^+(v) : v \in \mathcal{V}(H)\}$

$d^+(H), d^-(H)\}$. Note that

$\mathcal{V}(\widehat{H}), \mathcal{A}(\widehat{H}))$ with $\mathcal{V}(\widehat{H}) =$
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$$\mathcal{V}(LH) = \cup_{E \in \mathcal{E}(H)} \{(uEv) : u \in E^-, v \in E^+\}$$
$$\mathcal{E}(LH) = \cup_{v \in \mathcal{V}(H)} \{(EvF) : v \in E^+, v \in F^-\}$$

with $(EvF)^- = \{(wEv) : w \in E^-\}$ and $(EvF)^+ = \{(vFw) : w \in F^+\}$.

The iterated line digraph L^kH is defined by $L^kH = LL^{k-1}H$, with $L^0H = H$. Line digraphs iterations tend to increase the connectivities [1].

This paper concentrates on the relation between the connectivity, the diameter, and some new parameters, closely related with the parameter ℓ introduced by Fàbrega and Fiol [3]. There it is proved that if κ and λ denote respectively the vertex and arc-connectivity of a digraph of diameter D, then they are maximum where $D \leq 2l - 1$ and $D \leq 2l$, respectively. As a corollary, maximally connected iterated line digraphs are characterized.

In this paper some variations of parameter l, l_v and l_h, referred to hyperdigraphs, are introduced and we prove similar results:

Let $H = (\mathcal{V}(H), \mathcal{E}(H))$ be a hyperdigraph with minimum degree d, minimum size s, $\ell_v = \ell_v(H)$, $\ell_h = \ell_h(H)$, $D = D(H)$ and vertex and hyperarc connectivities κ and λ:

$$\kappa = d(\widehat{H}), \text{ if } D \leq 2\ell_v - 1 \text{ and } \lambda = d, \text{ if } D \leq 2\ell_h$$

and as corollary for iterated line hyperdigraphs:

$$\kappa(L^kH) = ds, \text{ if } k \geq D - 2\ell_v + 1 \text{ and } \lambda(L^kH) = d, \text{ if } k \geq D - 2\ell_h.$$

Also we state general bounds on the connectivity in terms of the some variations of the above parameters, ℓ_π^v and ℓ_π^h:

Let $H = (\mathcal{V}(H), \mathcal{E}(H))$ be a hyperdigraph with minimum degree d, minimum size s, $D = D(H)$ and vertex and hyperarc connectivities κ and λ:

- For any π, $0 \leq \pi \leq d(\widehat{H}) - 2$, $\kappa \geq d(\widehat{H}) - \pi$, if $D \leq 2\ell_\pi^v(H) - 1$;
- For any π, $0 \leq \pi \leq d - 2$, $\lambda \geq d - \pi$, if $D \leq 2\ell_\pi^h(H)$.

and the corresponding corollaries for iterated line digraphs:

- For any π, $0 \leq \pi \leq ds - 2$, $\kappa(L^kH) \geq ds$ if $k \leq D - 2\ell_\pi^v + 1$;
- For any π, $0 \leq \pi \leq d - 2$, $\lambda(L^kH) \geq d$ if $k \leq D - 2\ell_\pi^b$.

The terminology introduced is also shown to be useful to study the vulnerability. The *fault-diameter*, $\kappa(w, H)$, is the maximum diameter of the hyperdigraphs obtained by removing w vertices in H. Analogously, it is defined the *fault-diameter*, $\lambda(w, H)$, for hyperarcs.

Let $H = (\mathcal{V}(H), \mathcal{E}(H))$ be a hyperdigraph with minimum degree d, minimum size s, $D = D(H)$, $\ell_v = \ell_v(H)$ and $\ell_h = \ell_h(H)$.

- For any $k \geq D - 2\ell_v + 1$, $\kappa(w, L^k H) \leq D(L^k H) + C$, with $C = \max\{2(D - \ell_v), D + 1\}$ and for any $w = 1, \ldots, ds - 1$;
- For any $k \geq D - 2\ell_b$, $\lambda(w, L^k H) \leq D(L^k H) + C$, with $C = \max\{2(D - \ell_b), D + 1\}$, and for any $w = 1, \ldots, d - 1$.

The De Bruijn and Kautz bus networks were introduced in [2]. There it was shown that they have good order in relation to their maximum vertex degree and bus size. In fact, for digraphs they are generalizations of the best known families according to the aforementioned criteria.

Let $GB(d, n, s, m)$ and $GK(d, n, s, m)$ be a De Bruijn and Kautz hyperdigraphs with degree d, order n, size s and m hyperarcs, respectively. If $H = GK(d, n, s, m)$ or $H = GB(d, n, s, m)$, then

$$\kappa(H) \geq ds - 1, \text{ if } D(H) \geq 3 \text{ and } \lambda(H) \geq d - 1, \text{ if } D(H) \geq 2.$$

References

[1] J.C. Bermond, F. Ergincan, Directed Line Hypergraphs.

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[3] J. Fàbrega, M.A. Fiol, Maximally Connected Digraphs, *Journal of Graph Theory* **13-6** (1999), 657–688.