Electrical Power Networks, Combinatorial Optimization & Research Collaboration

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Electrical Power Networks

- Link sources of electrical power and consumers of electricity.
- Must be continuously monitored to prevent blackouts, power surges and overall, to guarantee certain quality of service.
- The state of a network is defined by the magnitude and phase angle of the electromagnetic wave at each and every one of its nodes and links.
- A Phasor Measurement Unit (PMU) measures magnitude and phase angle of the electromagnetic wave at the network location where it is placed.

Placing a PMU at each network location is unfeasible (costs) and unnecessary.

- A set of PMUs are placed at strategically selected network locations and their readings are synchronized via GPS (Global Positioning System).
- The PMU readings are then combined to calculate the magnitude and phase angle of the electricity wave at any network location without a PMU.

Monitoring an Electrical Power Network

- A PMU placement is feasible if the PMU readings are sufficient to determine the state of the network at any location without a PMU.
- A feasible PMU placement is optimal if it minimizes the number of PMUs.
- In electrical engineering, the PMU Placement Problem consists of finding an optimal PMU placement for a given power network.
- The Power Domination Problem was introduced in Graph Theory by Haynes, Hedetniemi, Hedetniemi & Henning so that:

Electrical Engineering		Graph Theory
electrical power network	-	graph
feasible PMU placement	-	power dominating set
optimal PMU placement	-	minimum power dominating set
optimal number of PMUs	-	power domination number
solution PMU placement problem	-	solution power domination problem

Power Domination

- Haynes, Hedetniemi, Hedetniemi & Henning (2002) introduced power domination in terms of observation rules for vertices and edges of a graph.
- Brueni & Heath (2005) obtained an equivalent definition of power domination only using observation rules for vertices.
- Aazami (2008) introduced discrete time intervals to study power domination.

Observation rules

Let $G = (V, E)$ be a graph and let S be an arbitrary set of vertices. For each positive integer t, the set of vertices observed by S at time t is $P^t[S]$ recursively defined by:

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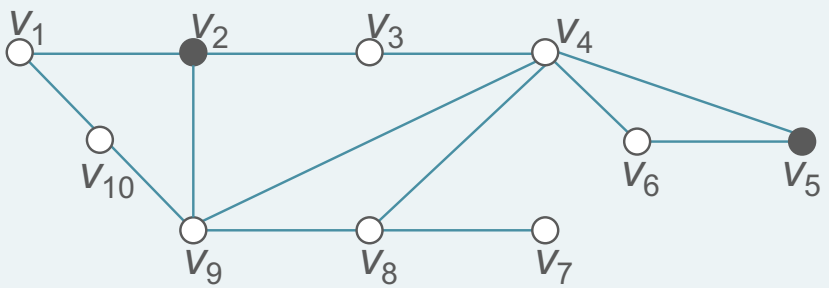

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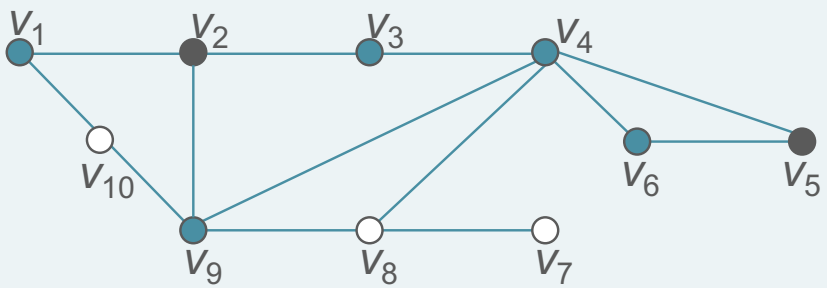

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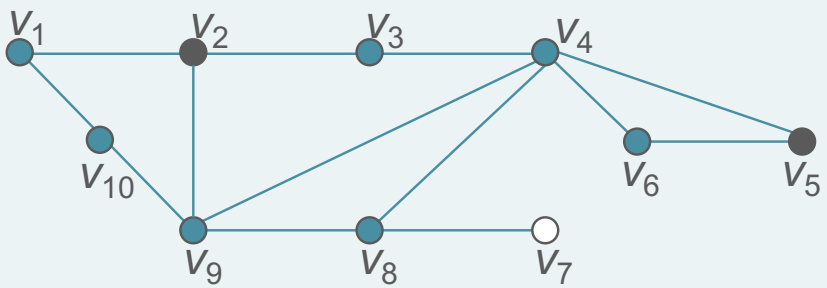

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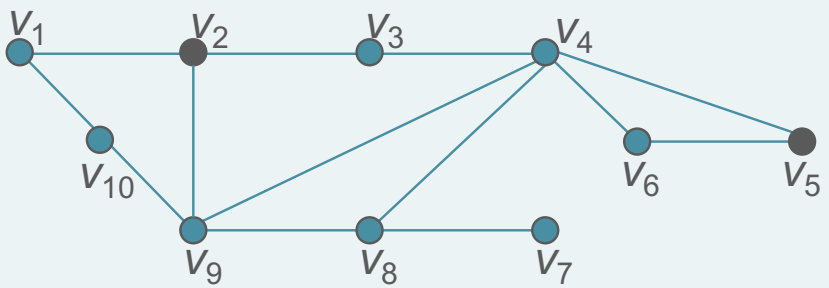

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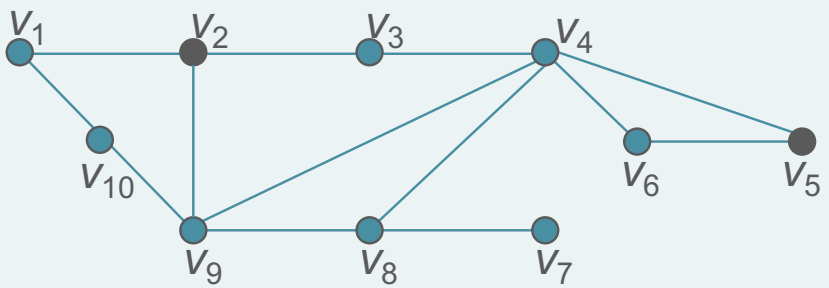

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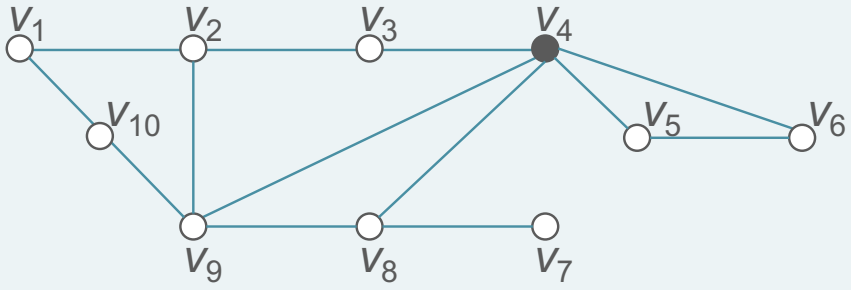

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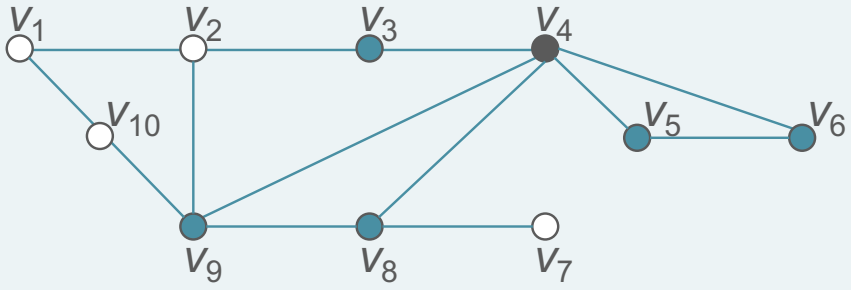

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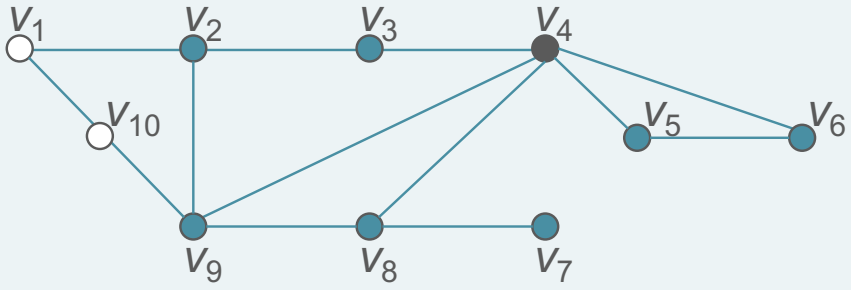

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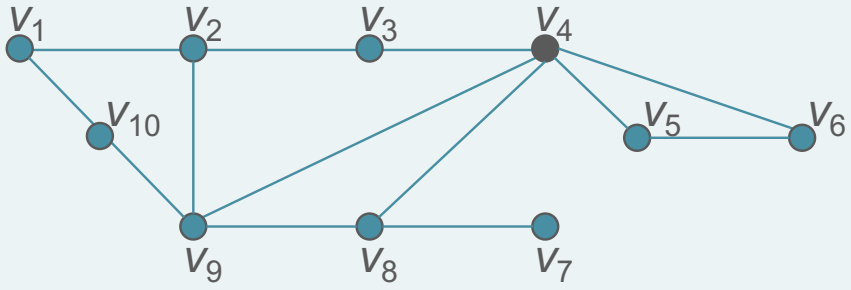

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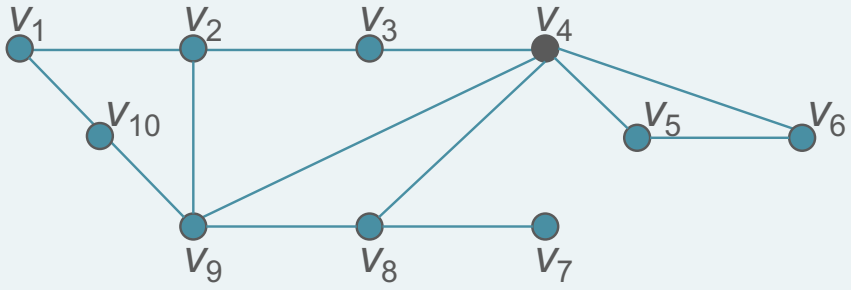

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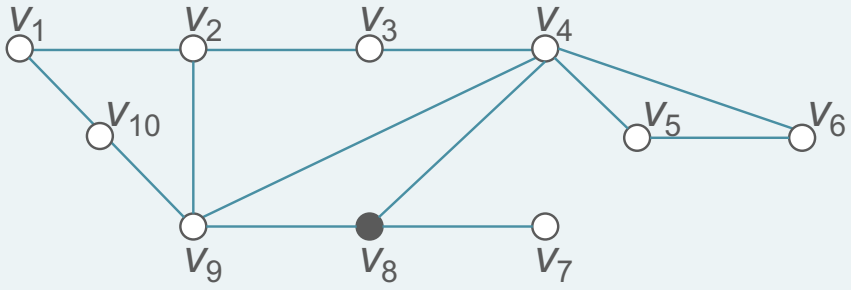

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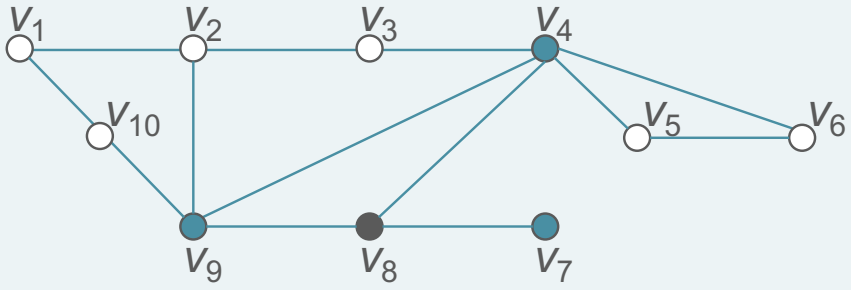

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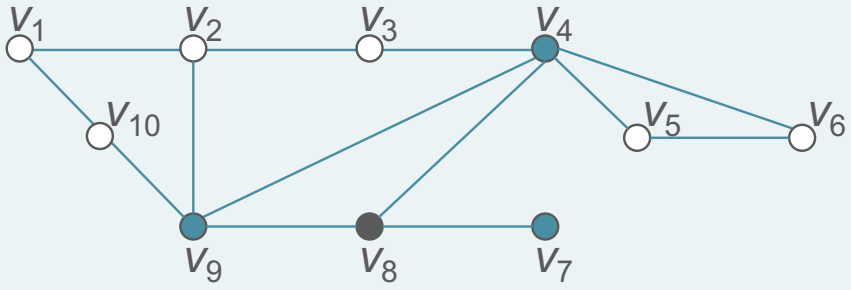

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$P^t[S] = P^1[S]$, $t > 1$
S not a power dominating set

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In any a graph G, for a given set of vertices S, there is a unique sequence $\{P^i[S]\}_{i \geq 1}$.

Power propagation time of a set

The power propagation time of a power dominating set S of $G = (V, E)$ is

$$\text{ppt}(G, S) = \min\{t \text{ integer} : P^t[S] = V\}.$$

$\text{ppt}(G, S)$ indicates the amount of data that must be transmitted when using the PMU placement defined by S to monitor the power network modeled by G.

Definitions

Power Dominating Set Problem (PDS)

Instance:	A graph G and a positive integer k
Question:	Does G have a power dominating set S such that $\lvert S\rvert \leq k$?

Minimum power dominating set

A minimum power dominating set (γ_P-set) is a power dominating set of minimum cardinality.

Power domination number

The power domination number of a graph G is $\gamma_P(G) = |S|$ where S is a γ_P-set of G.

Power propagation time of a graph

The power propagation time of graph G is $\mathrm{ppt}(G) = \min\{\mathrm{ppt}(G, S) : S\ \gamma_P\text{-set of } G\}$.

What is a name? The application to electrical power networks and the fact that $\mathrm{ppt}(G, S) = 1$ if and only if S is a dominating set of G.

Algorithmic Complexity

Theorem (Haynes, Hedetniemi, Hedetniemi & Henning, 2002)

The power domination problem (PDS) is NP-complete.

PDS is NP-complete

- Bipartite graphs *(Haynes, Hedetniemi, Hedetniemi & Henning, 2002)*
  Chordal graphs
- Split graphs *(Liao & Lee, 2005)*
- Planar graphs *(Guo, Niedermeier & Raible, 2008)*
  Circle graphs

PDS is polylogarithmic

- Interval graphs *(Liao & Lee, 2009)*
  Circular-arc graphs

PDS is linear

- Trees *(Haynes, Hedetniemi, Hedetniemi & Henning, 2002)*
- Interval graphs known interval order *(Liao & Lee 2005)*
- Block graphs *(Xu, Kang, Shan & Zhao, 2006)*

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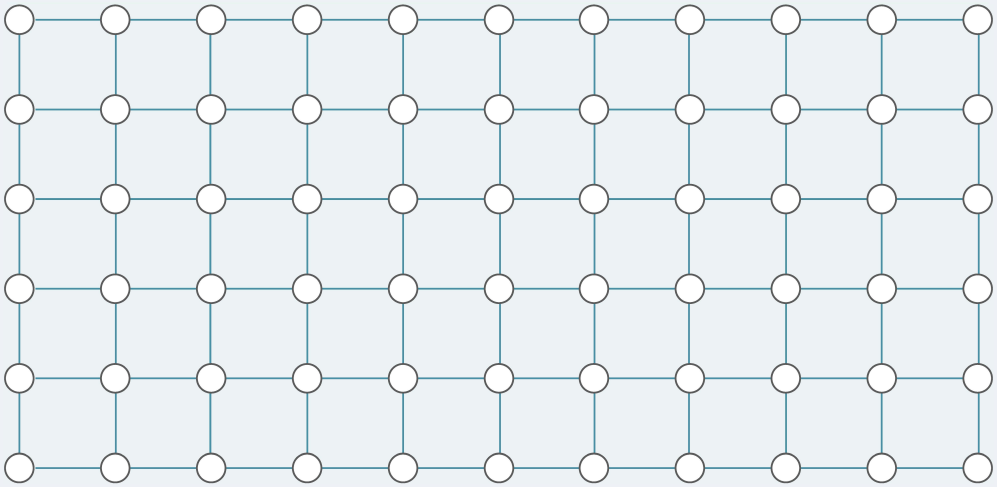

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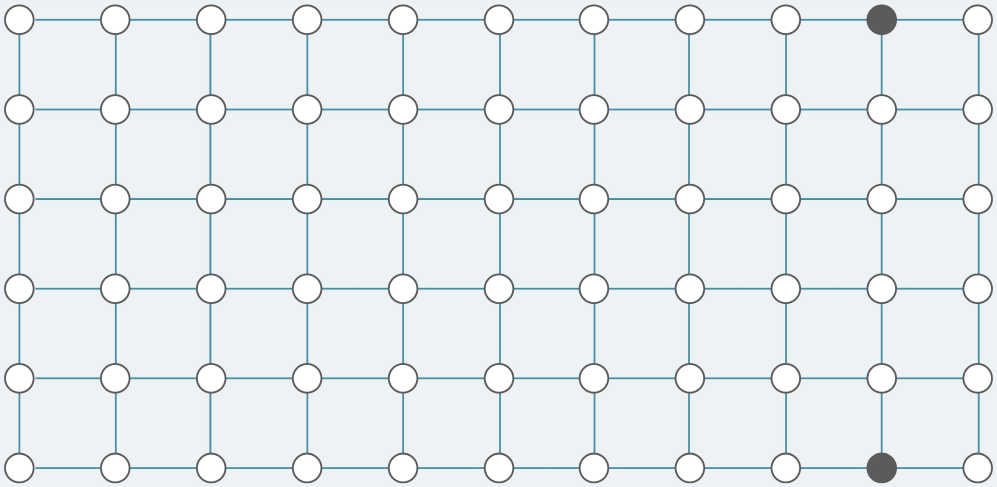

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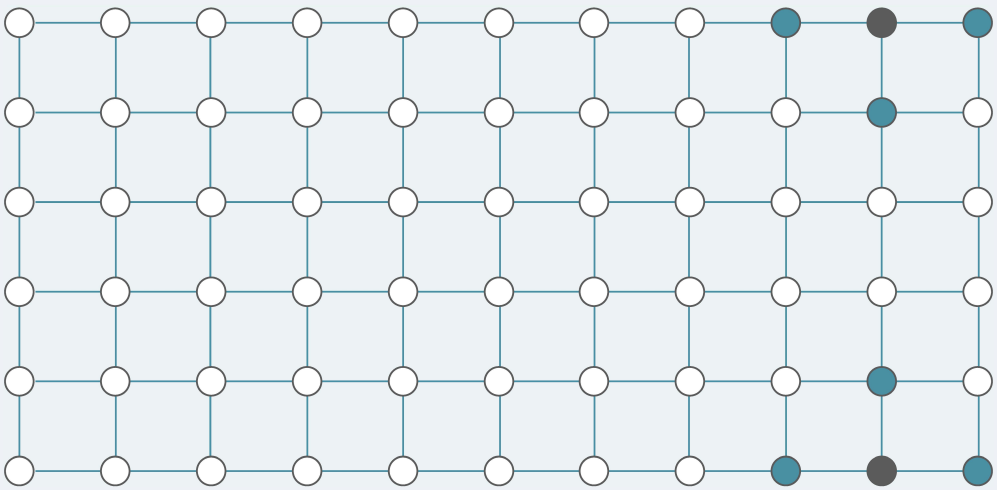

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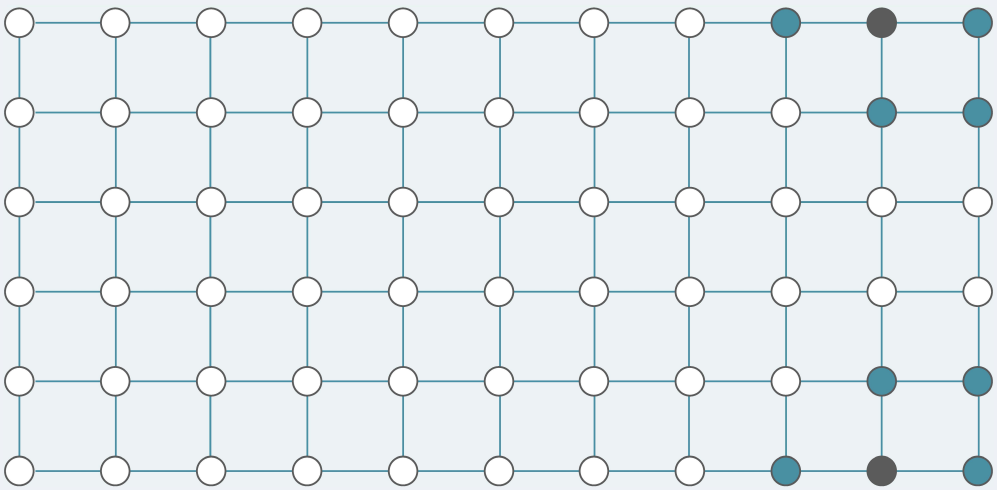

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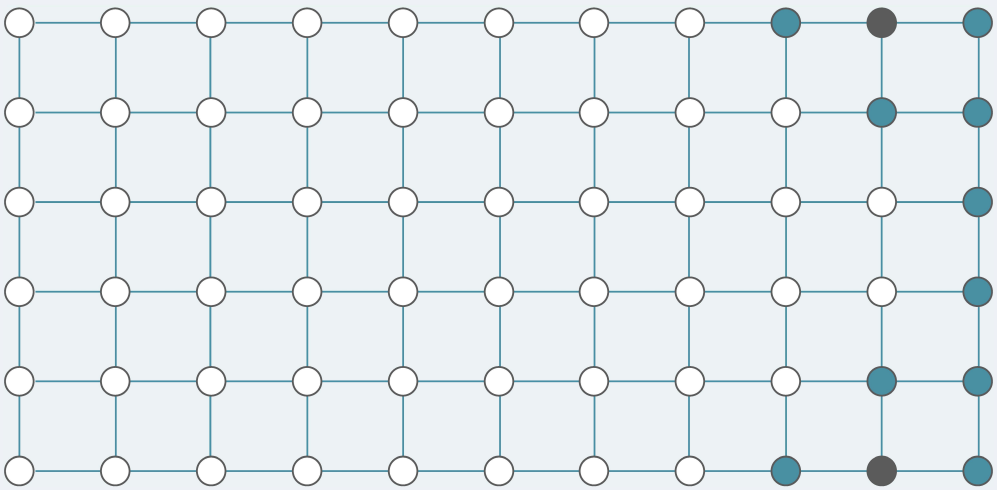

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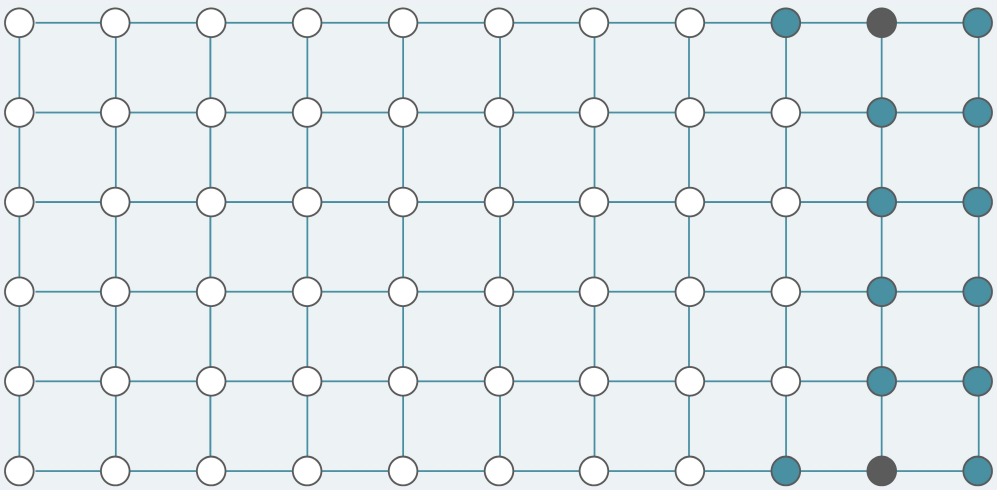

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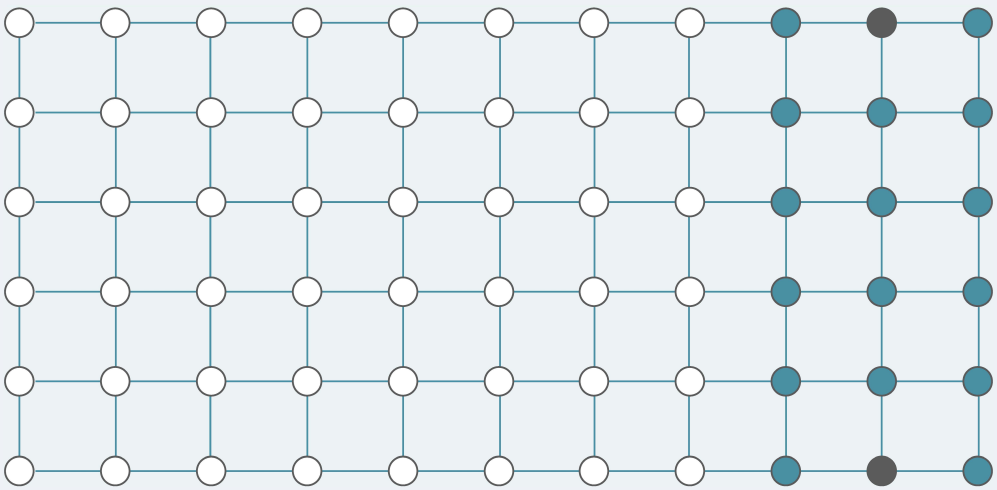

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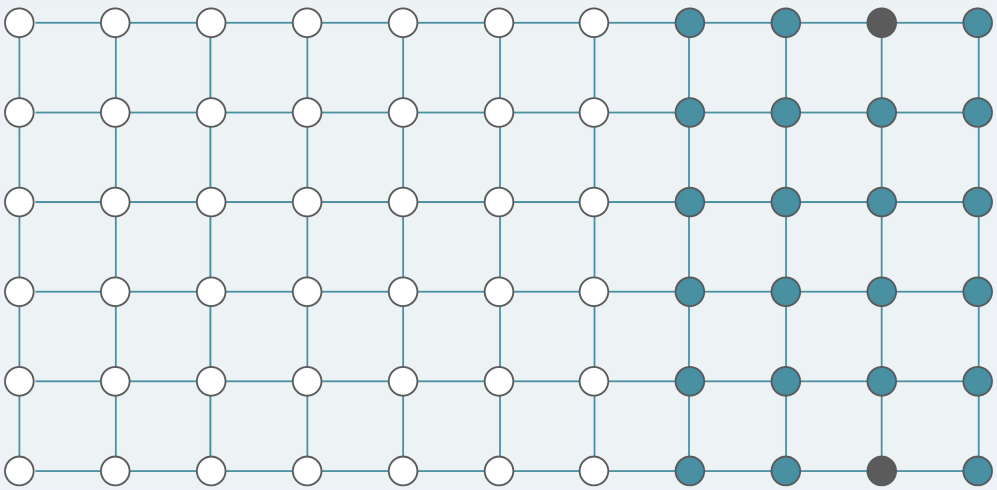

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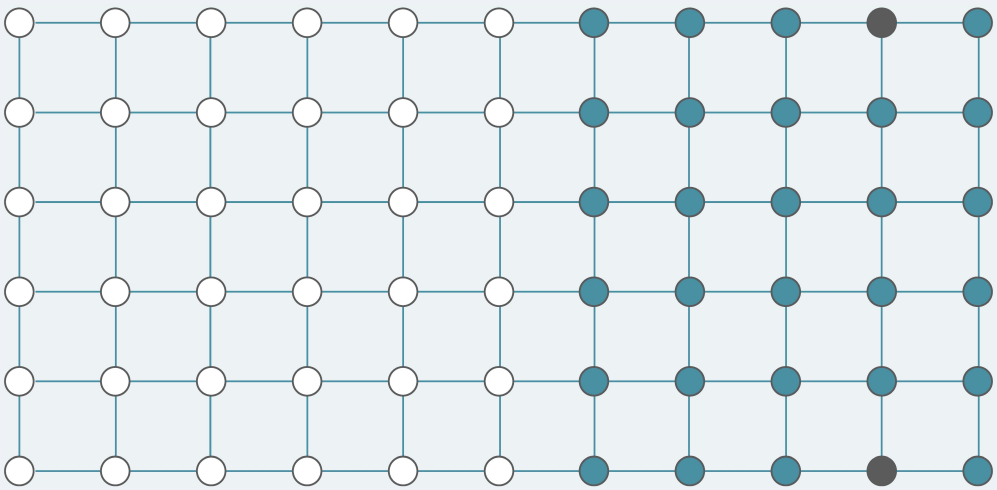

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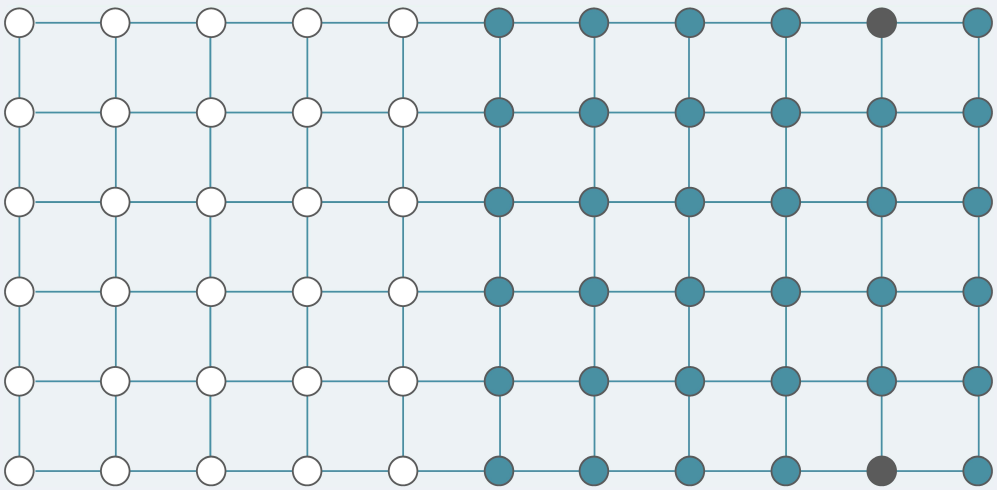

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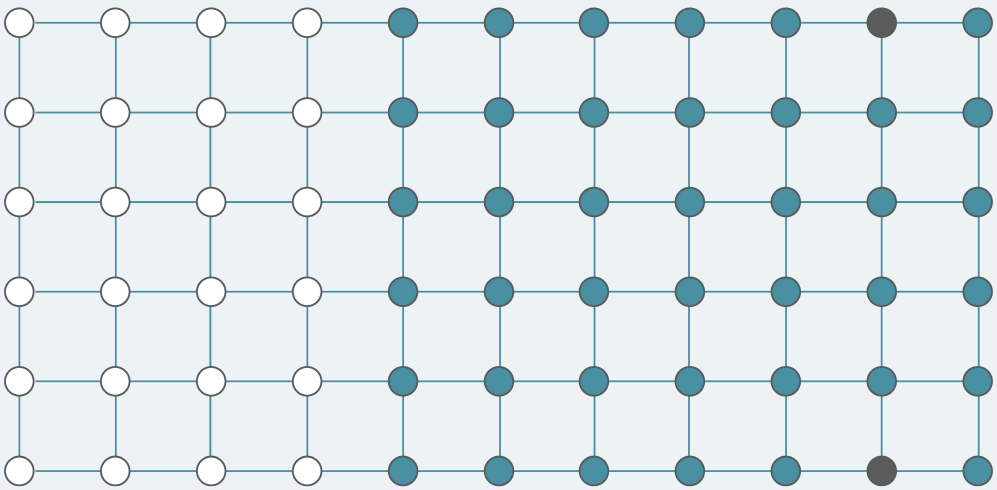

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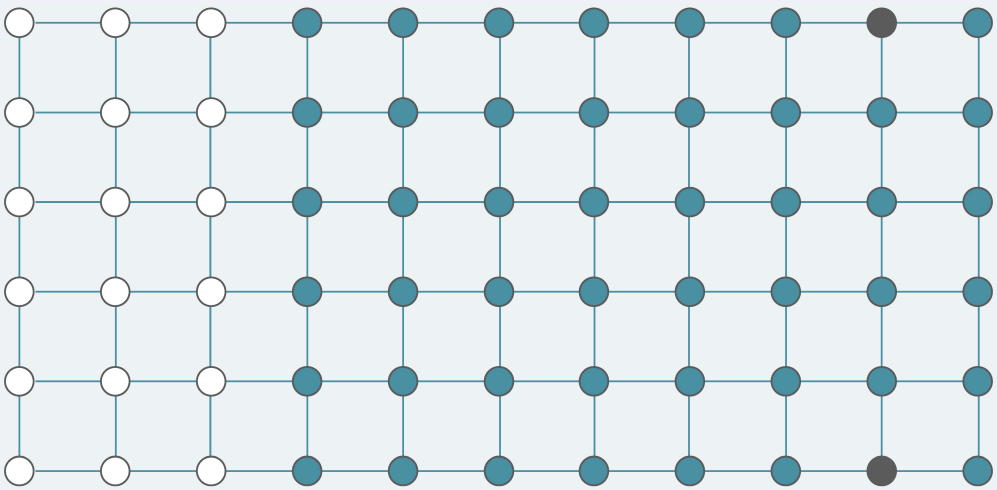

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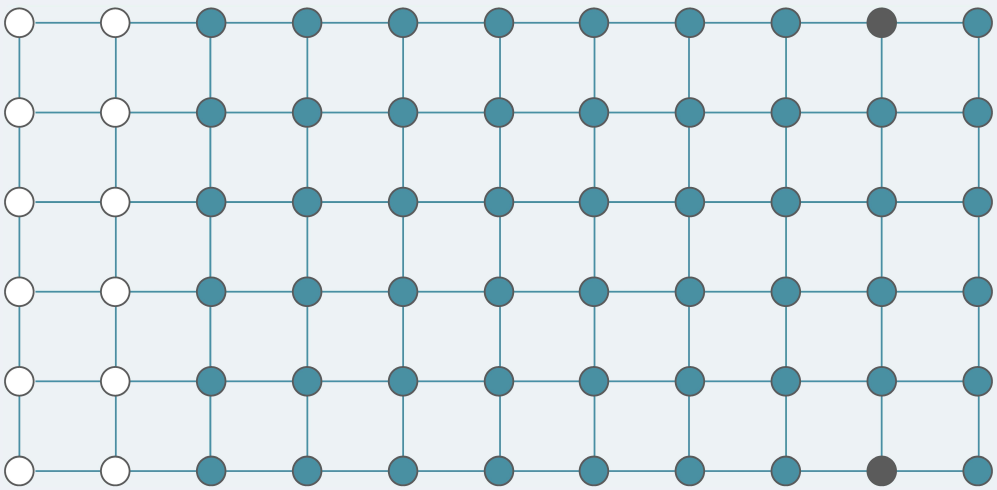

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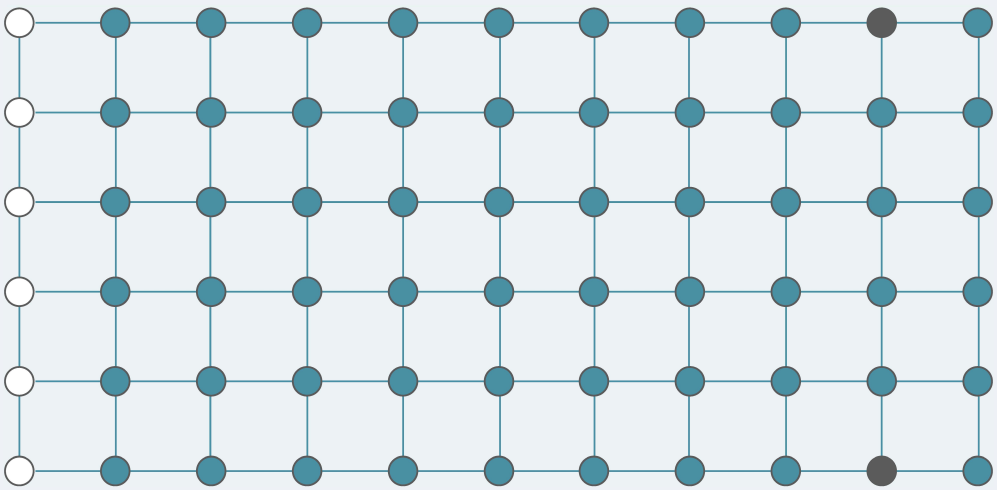

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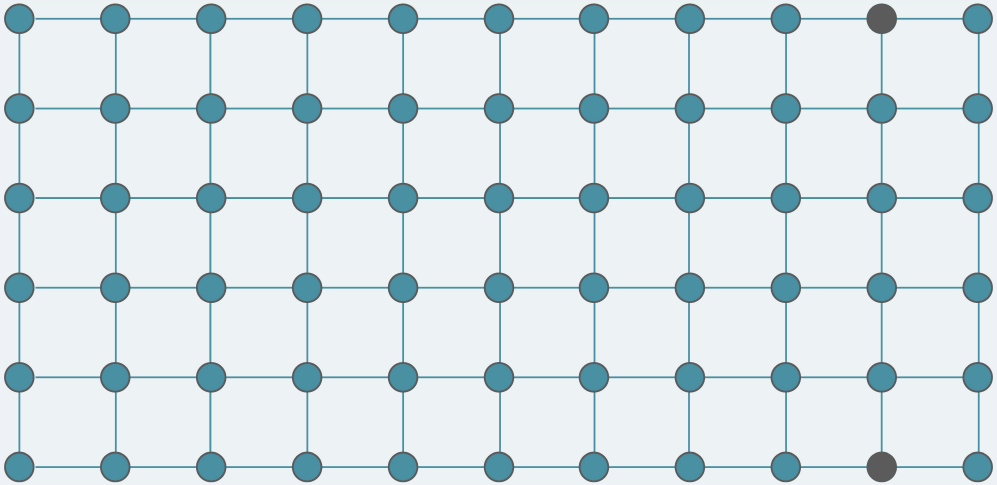

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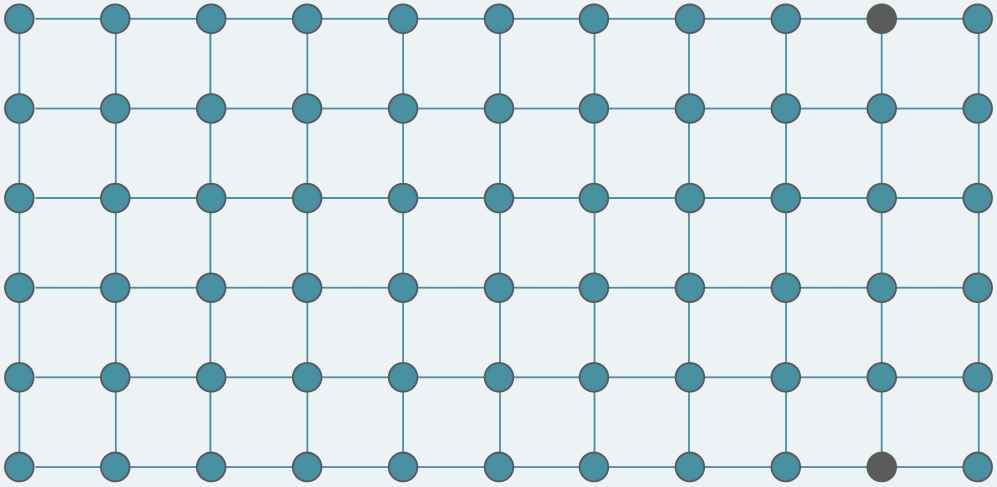

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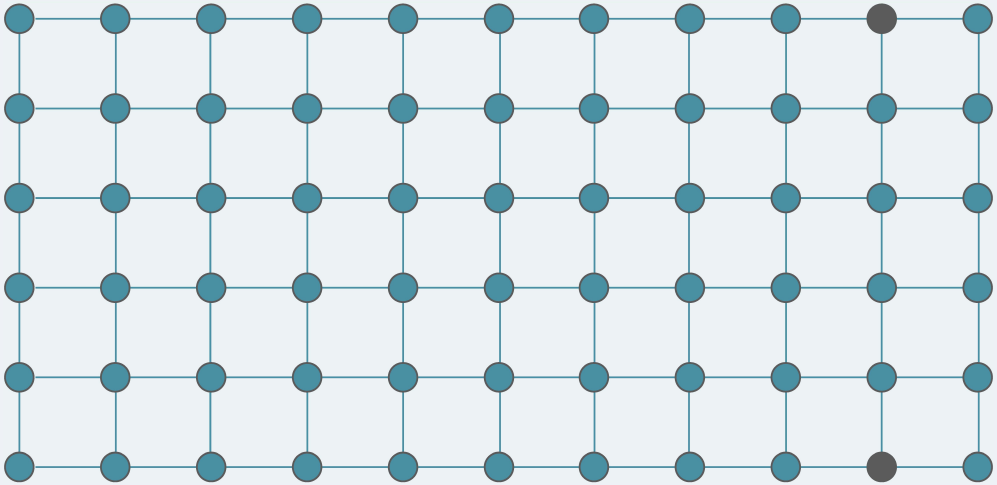

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$\gamma_P(P_n \times P_6) = 2$

Theorem (Dorfling & Henning, 2006)

For any integers $n \geq m \geq 3$, $\gamma_P(P_n \square P_m) = \begin{cases} \lceil \frac{m+1}{4} \rceil & \text{if } m \equiv 4 \bmod 8 \\ \lceil \frac{m}{4} \rceil & \text{otherwise} \end{cases}$

Power Domination

- In 2008, an undergraduate student taking my Honors Graph Theory course was interested on working on an undergraduate thesis.
- The Cartesian product of graphs G and H is $G\square H$ where $V(G\square H) = V(G) \times V(H)$ and $E(G\square H) = \{(x,x'),(y,y') : x = y, x'y' \in E(H), \text{ or } x' = y', xy \in E(G)\}$.
- If P_n and C_n are respectively the path and the cycle of order n, the rectangular $n \times m$ grid is $P_n\square P_m$, the cylinder of height n and girth m is $P_n\square C_m$ and the $n \times m$ torus is $C_n\square C_m$.

Theorem (Barrera & Ferrero, 2011)

For any integers $n \geq 1$ and $m \geq 3$, $\gamma_P(P_n\square C_m) \leq \min\{\lceil \frac{m+1}{4} \rceil, \lceil \frac{n+1}{2} \rceil\}$ and $\gamma_P(P_n\square C_m) = 2$ if $n = 2,3$ and $m \geq 4$ or if $n \geq 2$ and $4 \leq m \leq 7$.

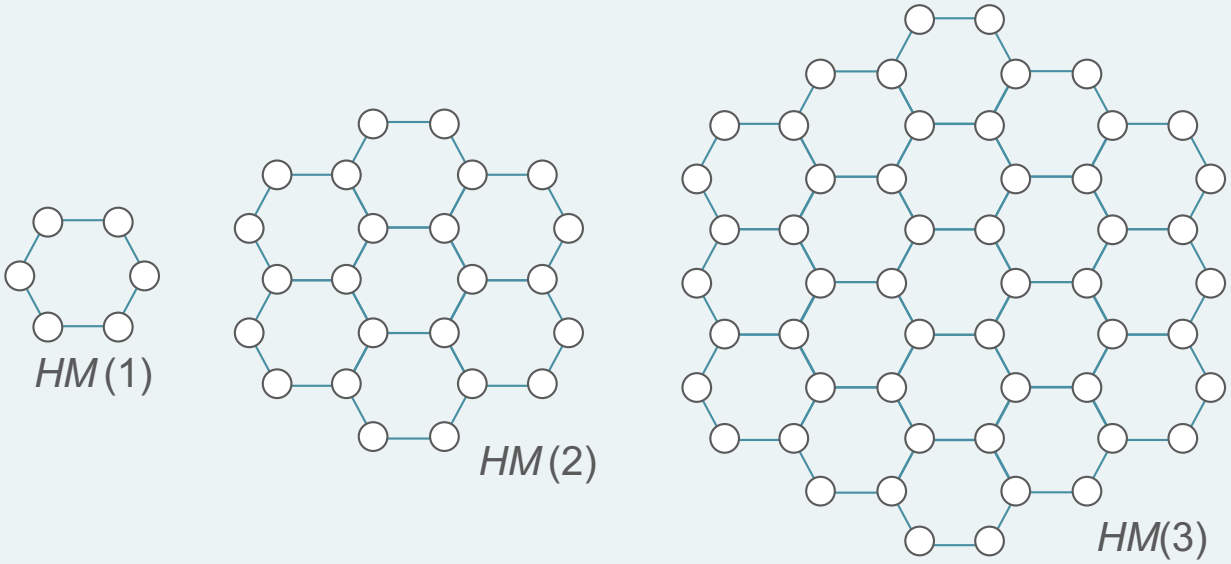
Theorem (Barrera & Ferrero, 2011)

For any integers $n \geq m \geq 3$, $\gamma_P(C_n\square C_m) \leq \begin{cases} \lceil \frac{m+1}{2} \rceil & \text{if } m \equiv 2 \bmod 4 \\ \lceil \frac{m}{2} \rceil & \text{otherwise} \end{cases}$

- Roberto Barrera, who now has Ph.D. in mathematics and is presently my colleague at Texas State University.

Power Domination

- In 2009, thanks to a sabbatical leave and an Erudite Fellowship, I had another opportunity to work on power domination.
- At Cochin University of Science & Technology in Kerala (India), I worked with Ambat Vijaykumar and Seema Varghese on honeycomb meshes.



Theorem (Ferrero, Varghese & Vijaykumar, 2011)

For any integer $n \geq 1$ the honeycomb mesh $HM(n)$ has $\gamma_P(HM(n)) = \lceil \frac{2n}{3} \rceil$.

Power Domination

- The results for rectangular grids, cylinders, tori and hexagonal meshes follow this pattern:

Given a graph family $\{G_n\}_{n \geq n_0}$ for some integer $n_0 > 0$,

1. Find minimum power dominating sets in a few of the smallest graphs of the given family $(G_{n_0}, G_{n_0+1}, \ldots)$ until observing a pattern for a good candidate to minimum power dominating set in a generic graph of the family. Let S_n be the set obtained by such pattern for a general graph G_n of the family.
2. Show S_n is a power dominating set of G_n to conclude $\gamma_P(G_n) \leq |S_n|$.
3. Use properties of the particular graph family to show $|S_n| \leq \gamma_P(G_n)$ and obtain $\gamma_P(G_n) = |S_n|$.

- In all the results for graph families, the first part of the proof (step 2) is easy but the second part (step 3) is usually long, technical and not elegant.
- The main reason for this kind of work was the lack of a lower bound on the power domination number of a graph.

Graphs & Matrices

Matrices of a graph

Let $G = (V, E)$ be a graph with $V = \{v_1, \ldots, v_n\}$. The set of matrices of G is $S(G)$, defined as the set of all real $n \times n$ symmetric matrices A such that for every $i \neq j$, $A_{i,j} \neq 0$ if v_i adjacent with v_j and $A_{i,j} = 0$ otherwise $(1 \leq i, j \leq n)$.

The adjacency matrix of G is the only matrix in $S(G)$ in which all entries are 0 or 1.

Minimum rank of a graph

The minimum rank of a graph G is $r(G) = min\{rank(A) : A \in S(G)$.

Maximum nullity of a graph

The maximum nullity of a graph G is $M(G) = max\{null(A) : A \in S(G)\}$.

- The rank minimization problem asks $r(G)$ for an arbitrary graph G and it is an important problem in Linear Algebra and in applications.
- Minimizing $r(G)$ is equivalent to maximizing $M(G)$ $(r(G) + M(G) = |G|)$.

Zero forcing

AIM Minimum Rank Special Graphs Work Group
F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I.Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, A. Wangness

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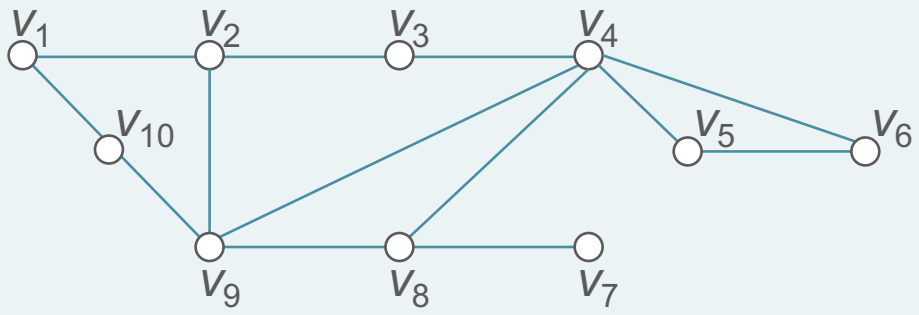
Example



$$S = \{v_3, v_4, v_5, v_9\}$$

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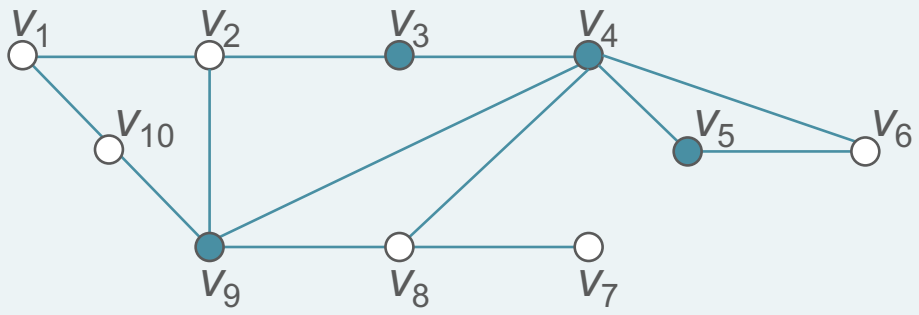

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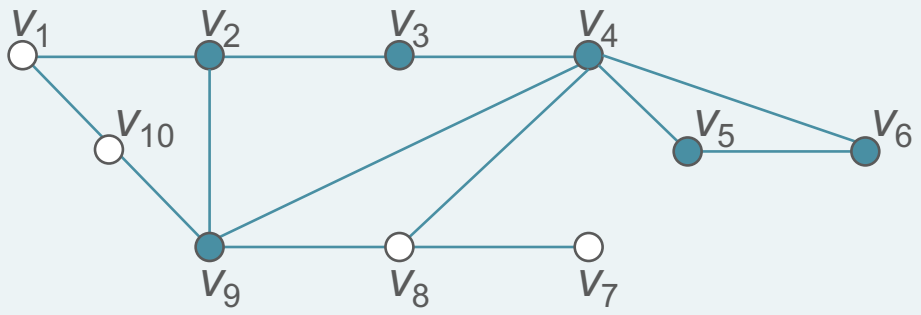

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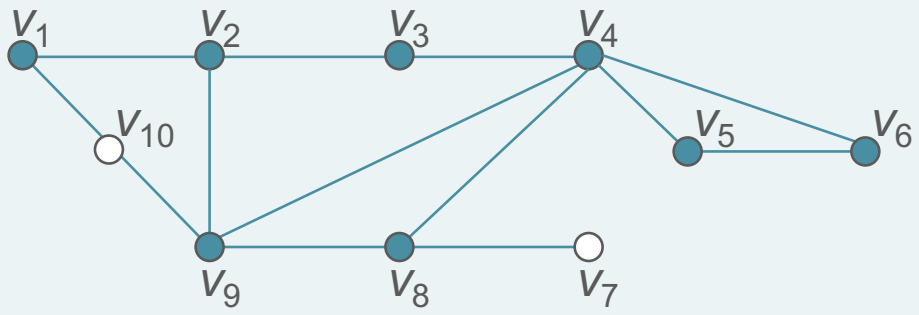

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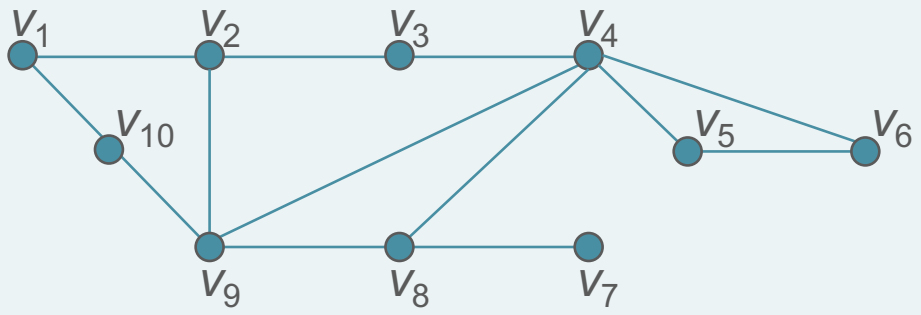

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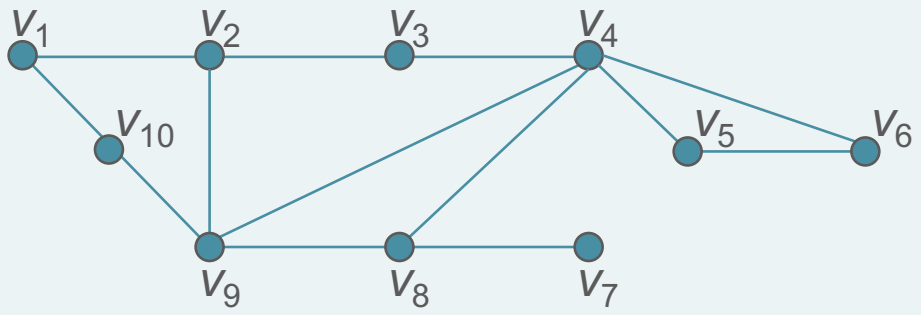

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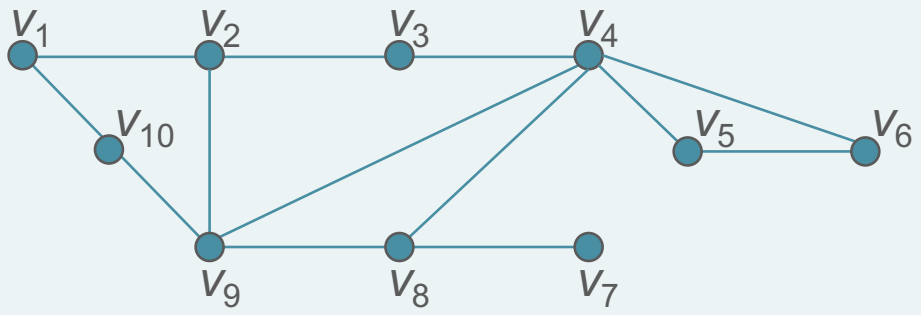

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Zero forcing

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- If $B_i[S]$ is the set of blue vertices after i iterations of the rule ($i \in \mathbb{Z}$, $i \geq 0$), then
  1. $B_0[S] = S$
  2. $B_{i+1}[S] = B_i[S] \cup \{u \in V \setminus B_i[S] : \exists v \in B_i[S], N(v) \setminus B_i[S] = \{u\}\}$

  and S is a zero forcing set of G if and only if $B_i[S] = V$ for some integer $i \geq 0$.

Observation rules

Let $G = (V, E)$ be a graph and let S be an arbitrary set of vertices. For each positive integer t, the set of vertices observed by S at time t is $P^t[S]$ recursively defined by:

1. $P^1[S] = N[S] = S \cup N(S)$
2. $P^{t+1}[S] = P^t[S] \cup \{u \in V \setminus P^t[S] : \exists v \in P^t[S], N(v) \setminus P^t[S] = \{u\}\}$

- In any graph G, for any $S \subseteq V$ and any integer $i \geq 0$: $B_i[N[S]] = P^{i+1}[S]$.

  Thus, $N[S]$ zero forcing set if and only if S is a power dominating set of G.

Zero forcing

Zero forcing was independently defined in quantum physics (graph infection) and in computer science (fast-mixed search).

Zero forcing set (AIM Minimum Rank Special Graphs Work Group, 2006)

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Minimum zero forcing set

A minimum zero forcing set is a zero forcing set of minimum cardinality.

Zero forcing number

A graph G with a minimum zero forcing set S has zero forcing number $\mathrm{Z}(G) = |S|$.

Theorem (AIM Minimum Rank Special Graphs Work Group, 2006)

In any graph G, $M(G) \leq \mathrm{Z}(G)$.

Power Domination & Zero Forcing

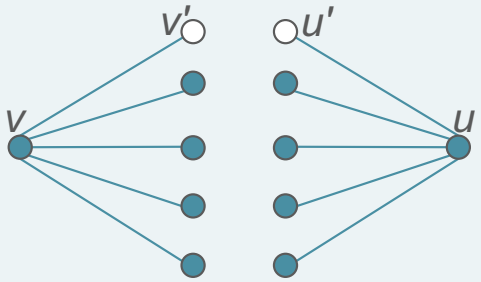
Observation: If G is a graph and S is a minimum power dominating set of G, then

- $N[S]$ is a zero forcing set of G, which implies $Z(G) \leq |N[S]|$
- If G has maximum degree Δ, then $|N[S]| \leq (\Delta + 1)|S|$ where $|S| = \gamma_P(G)$

Conclusion: $Z(G) \leq (\Delta + 1)\gamma_P(G)$ or equivalently, $\left\lceil \frac{Z(G)}{\Delta+1} \right\rceil \leq \gamma_P(G)$

Improvement:

Given $v \in S$ select a vertex $v' \in N(s)$
Then, $N[S] \setminus \{v'\}$ is a zero forcing set
Can we do the same for every $v \in S$?



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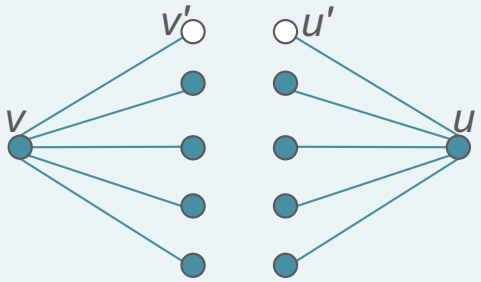
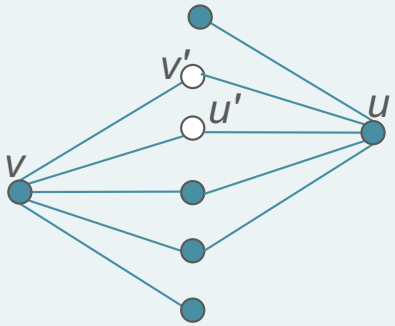
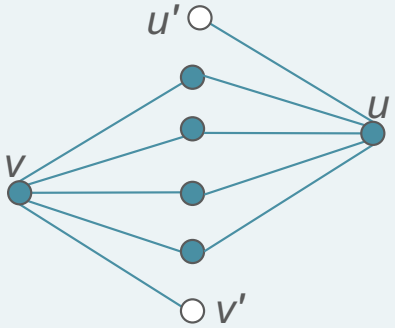


Yes, because S is a minimum power dominating set, so every vertex has a private neighbor in $N[S] \setminus S$





Power Domination & Zero Forcing

Theorem (Benson, Ferrero, Flagg, Furst, Hogben, Vasilevska, Wissman, 2016)

If G is a graph with maximum degree Δ then $\left\lceil \frac{Z(G)}{\Delta} \right\rceil \leq \gamma_P(G)$.

Theorem (AIM Minimum Rank Special Graphs Work Group, 2006)

In any graph G, $M(G) \leq Z(G)$.

Corollary (Benson, Ferrero, Flagg, Furst, Hogben, Vasilevska, Wissman, 2016)

Let G be a graph with maximum degree Δ. If $A \in S(G)$ then $\left\lceil \frac{null(A)}{\Delta} \right\rceil \leq \gamma_P(G)$.

This corollary, shortened the proofs of many known results and yielded new results on power domination and zero forcing for various graph products.

Power Domination & Zero Forcing

Some of the consequences of the Corollary by Benson, Ferrero, Flagg, Furst, Hogben, Vasilevska & Wissman with the support from REUF, AIM & ICERM.

Cartesian products

$$\text{If } n \geq m \geq 3 \text{ then } \gamma_P(C_m \square C_n) = \begin{cases} \frac{m}{2} + 1 & \text{if } m \equiv 2 \bmod 4 \\ \lceil \frac{m}{2} \rceil & \text{otherwise} \end{cases}$$

$$\gamma_P(P_m \square P_n) = \begin{cases} \lceil \frac{m+1}{4} \rceil & \text{if } m \equiv 4 \bmod 8 \\ \lceil \frac{m}{4} \rceil & \text{otherwise} \end{cases} .$$

$$\gamma_P(C_m \square P_n) = \min\{\lceil \tfrac{m+1}{4} \rceil, \lceil \tfrac{n+1}{2} \rceil\}$$

Tensor products

$$\text{If } n, m \geq 3 \text{ then } \gamma_P(P_m \times K_n) = \begin{cases} 2n - 1 & \text{if } m = n \text{ and } n \text{ is odd} \\ 2n & \text{otherwise} \end{cases}$$

$$\gamma_P(C_m \times K_n) = \begin{cases} 2n - 1 & \text{if } m = n \text{ and } n \text{ is odd} \\ 2n & \text{otherwise} \end{cases}$$

Lexicographic products

If $n \geq 2$ and $m \geq 3$ then $Z(K_n * C_m) = (n-1)m + 2$.

Other Topics

Benson, Ferrero, Flagg, Furst, Hogben & Vasilevska continue collaborating through 2019 thanks to the support received from REUF, AIM & ICERM.

Nordhaus-Gaddum problems

For a graph parameter ζ, the following are *Nordhaus-Gaddum* problems:

- Determine a (tight) lower or upper bound on $\zeta(G) + \zeta(\overline{G})$
- Determine a (tight) lower or upper bound on $\zeta(G) \cdot \zeta(\overline{G})$

Fault-tolerant power domination

Problem: Study power dominating sets that are resilient to failures and/or to the injection of corrupted information in the PMU system

We studied both problems for power domination and for zero forcing.

Generalizations

Definition (Chang, Dorbec, Montassier, Raspaud, 2012)

For an integer $k \geq 1$, a graph $G = (V, E)$ and a set $S \subseteq V$, define $P^1_{G,k}[S] = N[S]$ and $P^{i+1}_{G,k}[S] = P^i_{G,k}[S] \cup \{u : \exists v \in P^i_{G,k}[S], v \in N(u), |N(v) \setminus P^i_{G,k}[S]| \leq k\}, \forall i \geq 1$. If there exists t such that $P^t_{G,k}[S] = V$ then S is a k-power dominating set of G.

Definition (Amos, Caro, Davila, Pepper, 2015)

For an integer $k \geq 1$, a graph $G = (V, E)$ and a set $S \subseteq V$, define $F^1_{G,k}[S] = S$ and $F^{i+1}_{G,k}[S] = F^i_{G,k}[S] \cup \{u : \exists v \in F^i_{G,k}[S], v \in N(u), |N(v) \setminus F^i_{G,k}[S]| \leq k\}, \forall i \geq 1$. If there exists t such that $F^t_{G,k}[S] = V$ then S is a k-forcing set of G.

Theorem (Ferrero, Hogben, Kenter & Young, 2016)

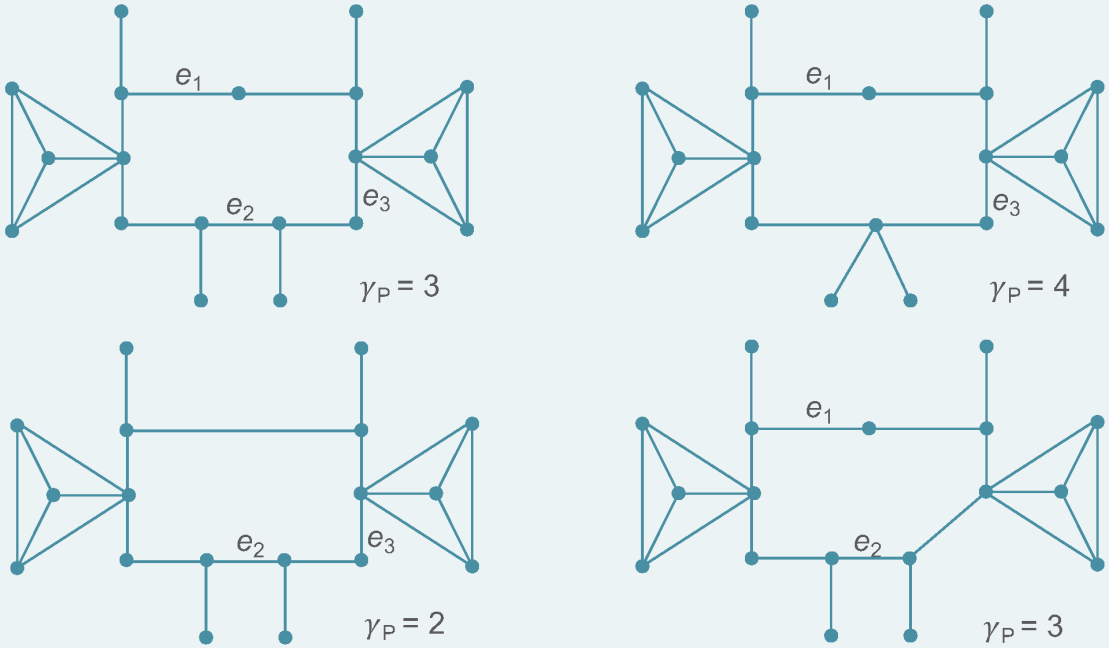
For an integer $k \geq 1$, let $\gamma_{P,k}(G)$ and $Z_k(G)$ denote the minimum cardinality of a k-power dominating set and a k-forcing set of G. If G is a connected graph of maximum degree $\Delta \geq k + 2$, then $Z_k(G) \leq \gamma_{P,k}(G)(\Delta + 1 - k)$.

Other Topics

Ferrero, Hogben, Kenter & Young continued their collaboration to study:

Edge Contraction & Subdivision

- Role of vertices of degree 2 in zero forcing and in power domination.
- Extension to vertices of degree less than k in k-forcing and k-power domination.



- Defined a new type of subgraph contraction to improve the computation of power dominating sets.

Other Topics

Ferrero, Hogben, Kenter & Young continued their collaboration to study:

Edge Contraction & Subdivision

- Role of vertices of degree 2 in zero forcing and in power domination.
- Extension to vertices of degree less than k in k-forcing and k-power domination.
- Defined a new type of subgraph contraction with important algorithmic implications for power domination.

Propagation time

- Defined propagation time for power domination i.e. $\mathrm{ppt}(G, S)$ and $\mathrm{ppt}(G)$.
- Proved that in every connected graph G of order n and maximum degree Δ,

$$\mathrm{ppt}(G) \geq \left\lceil \frac{n - \gamma_P(G)}{\Delta \gamma_P(G)} \right\rceil$$

Restricted Power Domination

C. Bozeman, B. Brimkov, C. Erickson, D. Ferrero, M. Flagg & L. Hogben, during an AIM Research Collaboration Workshop in 2017, started studying the following problem.

Question

What is the minimum number of additional PMUs needed to monitor an electrical power network that has been expanded, if the existing PMUs remain in place?

Definition

Let $G = (V, E)$ be a graph and let $X \subseteq V$. A set $S \subseteq V(G)$ is a *power dominating set of G subject to X* if S is a power dominating set of G and $X \subseteq S$. The *restricted power domination number of G subject to X* is the minimum number of vertices in a power dominating set that contains X, and is denoted by $\gamma_P(G; X)$.

Results

- Tight bounds on $\gamma_P(G; X)$
- Exact results and algorithms to find $\gamma_P(G; X)$
- Linear time algorithm to find $\gamma_P(G; X)$ if G is a graph with bounded treewidth.

Linkages & Zero Forcing

During the same AIM Research Collaboration Workshop in 2017, D. Ferrero, M. Flagg, T. Hall, L. Hogben, J. Lin, S. Meyer, S. Nasserasr & B. Shader started collaborating on another problem.

Question

In a graph G with a zero forcing set S, as vertices force color changes in their neighbors, some paths are being defined. How can these paths be described?

Results

- Proved that spanning forcing paths of a zero forcing process form a spanning rigid linkage, which are a special type of vital linkages, a graph theory concept introduced by Robertson & Seymour in the sequence of papers leading to the proof of the Perfect Graph Theorem.
- Showed that a particular type of rigid linkages provide bounds on the multiplicities of other eigenvalues of the family of matrices described by a graph.

Product Throttling

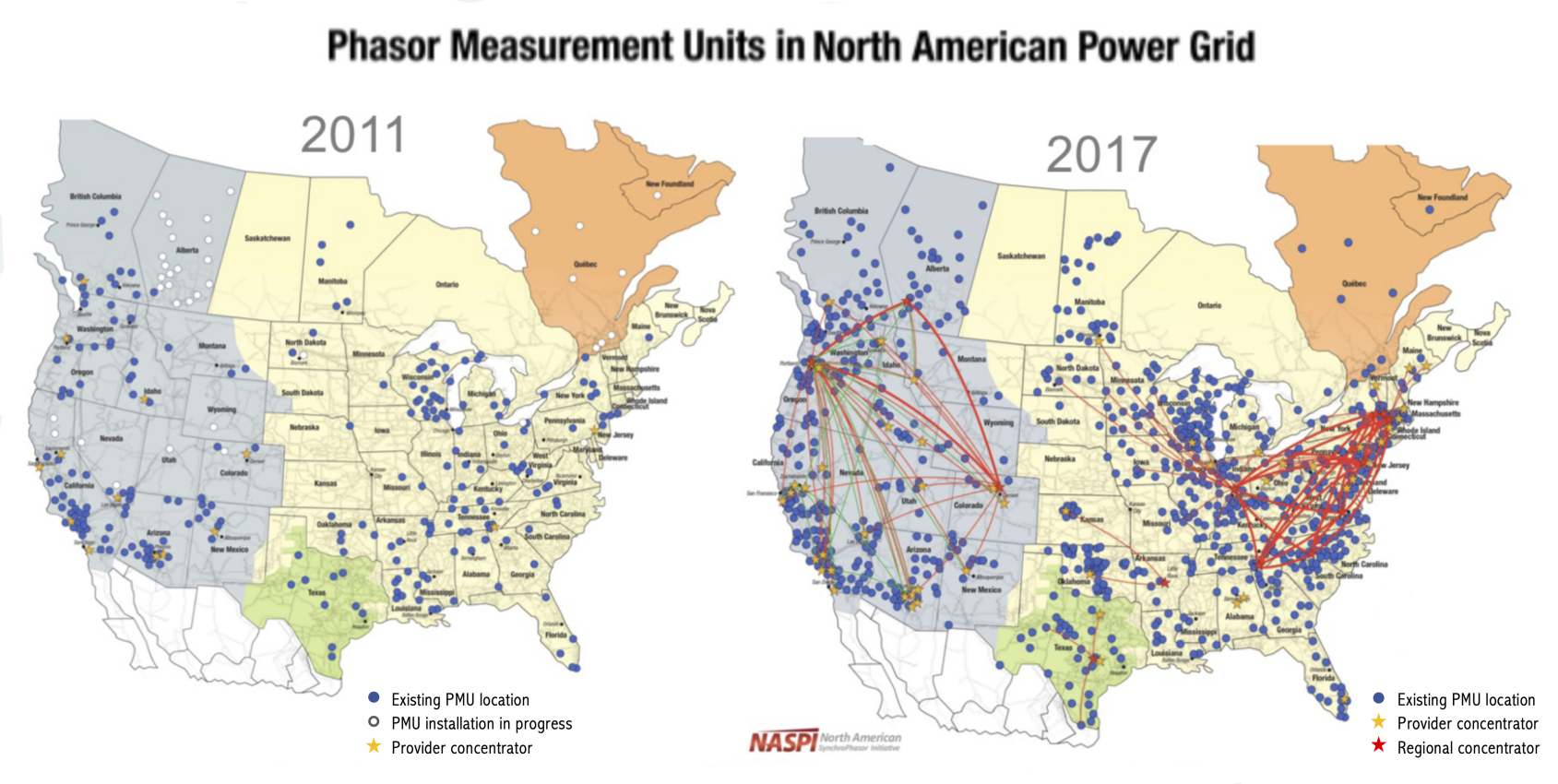
S. Anderson, K.Collins, D. Ferrero, L. Hogben, C. Mayer, A. Trenk & S. Walker started a collaboration during the Workshop for Research in Graph Theory and Applications held at IMA in 2019.

Problem

Observation of large scale PMU systems and new technological developments shows:

- Minimizing the number of PMUs produces unsatisfactory results, mainly due to the lack of redundancy to recover data lost in the event of failures.
- The addition of even a few additional PMUs produces so many advantages that offsets the increase in cost.

PMU system in North America



The large scale deployment of wide-area PMU started in 2008 and expanded very quickly.

Product Throttling

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Problem

Observation of large scale PMU systems and new technological developments shows:

- Minimizing the number of PMUs produces unsatisfactory results, mainly due to the lack of redundancy to recover data lost in the event of failures.
- The addition of even a few additional PMUs produces so many advantages that offsets the increase in cost.
- In graph searching there is always a compromise between number of searchers and search time, and throttling means minimizing their sum or their product.

Results

- Extreme values of $\mathrm{th}^{\times}_{\gamma_P}(G)$
- Determined $\gamma_P(G)$ and $\mathrm{th}^{\times}_{\gamma_P}(G)$ for unit interval graphs
- Bounds and exact values for $\mathrm{th}^{\times}_{\gamma_P}(G \square H)$

Reconfiguration

Research by B.Bjorkman, C.Bozeman, D.Ferrero, M.Flagg, C.Grood, L.Hogben, B.Jacobs & C.Reinhart, within the framework of the AIM Research Community on Inverse Eigenvalue Problems for Graphs.

Reconfiguration

- The reconfiguration version of a problem asks if it is possible to transform a feasible solution to the problem into another one, by iteratively applying certain operation guaranteeing that each intermediate step is also a feasible solution.
- Vertices of a reconfiguration graph correspond to feasible solutions and edges join two solutions that can be obtain from each other by applying the operation once.
- If the feasible solutions to P are sets, typical operations are token addition and removal (TAR) and token exchange (TE).

Results

- Properties of the TAR and the TE reconfiguration graphs for power dominating sets.

The End

Conclusion

An application to a program at any of the Math Institutes does not require much effort and there is a lot to gain if you receive their support.

Thank you!

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